M E T U
Northern Cyprus Campus

Math 260 Linear Algebra Final Exam 24.01.2014		
Last Name Name : Student No KEY	Dept./Sec.: Time : 09: 00 Duration : 80 *minutes*	Signature
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

1	2	3	4	5				

Q1 (20=10+10 p.) Consider the vectors **b**: $b_1 = (1,1,0,0), b_2 = (0,1,-1,0), b_3 = (0,0,1,1)$ in $\mathbb{R}^4$. **a)** Find the subspace $W = \mathrm{Span}(\mathbf{b})$ spanned by these vectors in $\mathbb{R}^4$.

$\lambda b_1 + \mu b_2 + \theta b_3 = (\lambda, \lambda+\mu, -\mu+\theta, \theta)$. Hence

$$\left.\begin{array}{r} \lambda = x \\ \lambda+\mu = y \\ -\mu+\theta = z \\ w = \theta \end{array}\right\} \Rightarrow \begin{array}{l} \mu = y-x \\ z = -(y-x)+w \Rightarrow -x+y+z-w=0. \text{ Thus} \\ W = \{-x+y+z-w=0\}. \end{array}$$

b) Using the basis **b** for W and Gram-Schmidt orthonormalization process, find an orthonormal basis **a** for W.

Put $a_1 = b_1 = (1,1,0,0)$,

$$a_2 = b_2 - \frac{\langle b_2, a_1\rangle}{\langle a_1, a_1\rangle} a_1 = (0,1,-1,0) - \frac{1}{2}(1,1,0,0) = (-1/2, 1/2, -1, 0),$$

$$a_3 = b_3 - \frac{\langle b_3, a_1\rangle}{\langle a_1, a_1\rangle} a_1 - \frac{\langle b_3, a_2\rangle}{\langle a_2, a_2\rangle} a_2 = (0,0,1,1) - \frac{0}{2} a_1 +$$
$$+ \frac{2}{3}(-1/2, 1/2, -1, 0) = (-1/3, 1/3, 1/3, 1)$$

Finally, we have $\frac{a_1}{\|a_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$,

$$\frac{a_2}{\|a_2\|} = \sqrt{\frac{2}{3}}\left(-1/2, 1/2, -1, 0\right), \quad \frac{a_3}{\|a_3\|} = \frac{\sqrt{3}}{2}\left(-1/3, 1/3, 1/3, 1\right).$$

Q2 (20 p.) Diagonalize the matrix $A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$ if it is possible.

Note that $\Delta_A(t) = |A - t\,I| = \begin{vmatrix} 4-t & 6 & 0 \\ -3 & -5-t & 0 \\ -3 & -6 & 1-t \end{vmatrix} =$

$= (1-t)\begin{vmatrix} 4-t & 6 \\ -3 & -5-t \end{vmatrix} = -(t-1)(t^2+t-2) = -(t-1)^2(t+2)$, that is,

$\sigma(A) = \{1^{(2)}, -2^{(1)}\}$.

$\lambda = 1$, $A - 1 = \begin{bmatrix} 3 & 6 & 0 \\ -3 & -6 & 0 \\ -3 & -6 & 0 \end{bmatrix}$, $V_1 = \ker(A-1) = \{x + 2y = 0\}$, $m(1) = 2$.

Put $f_1 = (2, -1, 0)$, $f_2 = (0, 0, 1)$ is a basis for V_1.

$\lambda = -2$, $A + 2 = \begin{bmatrix} 6 & 6 & 0 \\ -3 & -3 & 0 \\ -3 & -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $V_{-2} = \ker(A+2) = \{x = -z = -y\} =$

$= \operatorname{Span}\{(1, -1, -1)\}$

Hence $f = ((2,-1,0), (0,0,1), (1,-1,-1))$ is a Jordan basis for A and $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Q3 (10 p.) Consider the inner-product space $V = M_2(\mathbb{C})$ with the inner-product $\langle A, B \rangle = \operatorname{tr}(AB^*)$. Find the norm $\|A\|$ of the matrix $A = \begin{bmatrix} 1 & i \\ 2 & -i \end{bmatrix}$.

Since $A^* = \begin{bmatrix} 1 & 2 \\ -i & i \end{bmatrix}$, we derive that $\|A\|^2 = \operatorname{tr}(AA^*)$

$= \operatorname{tr}\begin{bmatrix} 1 & i \\ 2 & -i \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -i & i \end{bmatrix} = \operatorname{tr}\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} = 7$, that is,

$\|A\| = \sqrt{7}$.

Q4 **(25 p.)** Consider the following linear homogeneous system
$$\begin{cases} -x_1 + x_2 + x_3 + x_4 - x_5 &= 0 \\ -2x_1 + x_2 + x_4 &= 0 \\ 2x_4 + x_5 - x_6 &= 0 \\ 2x_2 + 4x_4 + x_6 &= 0 \end{cases}$$
. Find a basis $\mathbf{f}$ for the subspace Sol in $\mathbb{R}^6$.

$$\begin{array}{c} \\ R_2 - 2R_1 \\ \\ R_3 \leftrightarrow R_4 \end{array} \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -2 & -1 & 2 & 0 \\ 0 & 2 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix} \begin{array}{c} \\ \\ R_3 + 2R_2 \\ \\ \end{array} \sim$$

$$\begin{array}{c} -(R_1 + R_2) \\ -R_2 \\ \\ \\ \end{array} \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -2 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & -4 & 2 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 4 & 0 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 & 1 \\ 0 & 0 & -4 & 2 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 & 0 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 3 \\ 0 & 0 & -4 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix}$$
, which is a reduced echelon form of the system. Thus

$4x_1 = x_5 - 2x_6$, $2x_2 = 2x_5 - 3x_6$, $4x_3 = -3x_5 - 2x_6$, $2x_4 = -x_5 + x_6$.

In particular, $f_1 = (-1, -3, 1, 1, 0, 2)$, $f_2 = (1, 4, 3, -2, 4, 0)$ are linearly independent vectors from Sol, that is, $Sol = \mathrm{Span}\{f_1, f_2\}$, $\dim(Sol) = 2$.

Q5 (25 p.) Find the Jordan basis $\mathbf{f} = (f_1, f_2, f_3, f_4, f_5, f_6)$ and related Jordan matrix J of the following linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$,

$$T(\mathbf{x}) = \left(\underbrace{-2x_1 + x_3 - x_4}, \underbrace{-2x_2}, \underbrace{-2x_3 + x_6}, \underbrace{-2x_4}, \underbrace{-x_2 - 2x_5 + 3x_6}, \underbrace{2x_4 - 3x_6}\right),$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$.

Note that $M_{(e,e)}(T) = \begin{bmatrix} -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 2 & 0 & -3 \end{bmatrix}$, $\Delta(t) = (t+2)^5(t+3)$, $\sigma(T) = \{-2^{(5)}, -3^{(1)}\}$.

1) $\lambda = -2$, $(T+2)(\vec{x}) = (x_3 - x_4, 0, x_6, 0, -x_2 + 3x_6, 2x_4 - x_6)$,

$V_{-2,1} = \ker(T+2) = \{x_2 = x_3 = x_4 = x_6 = 0\}$, $m(-2) = 2 < 5 = \mathrm{alg}(-2)$,

$(T+2)^2(\vec{x}) = (x_6, 0, 2x_4 - x_6, 0, 6x_4 - 3x_6, -2x_4 + x_6)$,

$V_{-2,2} = \ker(T+2)^2 = \{x_4 = x_6 = 0\}$, $\dim(V_{-2,2}/V_{-2,1}) = 2$,

$(T+2)^3(\vec{x}) = (2x_4 - x_6, 0, -2x_4 + x_6, 0, -6x_4 + 3x_6, 2x_4 - x_6)$,

$V_{-2,3} = \ker(T+3)^3 = \{2x_4 = x_6\}$, $\dim(V_{-2,3}/V_{-2,2}) = 1$.

Whence $\overline{V_{-2}} = V_{-2,3}$. Choose

1	f_1		$V_{-2,3}/V_{-2,2}$
2	f_2	f_4	$V_{-2,2}/V_{-2,1}$
2	f_3	f_5	$V_{-2,1}$
	$\mathcal{U}_1 \oplus \mathcal{U}_2$		

$f_1 = (0, 0, 0, 1, 0, 2) \in V_{-2,3}/V_{-2,2}$,

$f_2 = (T+2)f_1 = (-1, 0, 2, 0, 6, 0)$, $f_4 = (0, 1, 0, 0, 0, 0)$

$f_3 = (T+2)^2 f_1 = (2, 0, 0, 0, 0, 0)$, $f_5 = (T+2)f_4 = (0, 0, 0, 0, -1, 0)$.

So, $\mathcal{U}_1 = \mathrm{Span}\{f_1, f_2, f_3\}$, $\mathcal{U}_2 = \mathrm{Span}\{f_4, f_5\}$

2) $\lambda = -3$, $(T+3)(\vec{x}) = (x_1 + x_3 - x_4, x_2, x_3 + x_6, x_4, -x_2 + x_5 + 3x_6, 2x_4)$,

$V_{-3,1} = \ker(T+3) = \{x_2 = x_4 = 0,\ x_1 = -x_3 = x_6 = -\frac{1}{3}x_5\}$,

$f_6 = (-1, 0, 1, 0, 3, -1)$. Hence

$f = (f_1, f_2, f_3, f_4, f_5, f_6)$ is a Jordan basis for T, and

$$J = \begin{bmatrix} -2 & 0 & 0 & & & \\ 1 & -2 & 0 & & 0 & \\ 0 & 1 & -2 & & & \\ & & & -2 & 0 & \\ & 0 & & 1 & -2 & \\ & & & & & -3 \end{bmatrix}$$