M E T U
Northern Cyprus Campus

Math 260 Linear Algebra Midterm Exam I 06.11.2012		
Last Name : Name : Student No :	Dept./Sec.: Time : 17: 40 Duration : 80 *minutes*	Signature
5 QUESTIONS ON 4 PAGES	KEY	TOTAL 100 POINTS

1	2	3	4	5				

Q1 (10 p.) Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). Prove the following elementary assertions:

a) The space V admits only one zero element 0_V

If there were two different zero's 0_V and $0_V'$, we would have

$$0_V' = 0_V' + 0_V = 0_V + 0_V' = 0_V,$$

a contradiction.

b) Each element $v \in V$ admits only one opposite element $-v \in V$.

Assume v has two opposite elements x and y, so $v + x = v + y = 0_V$. Then

$$x = x + 0_V = x + (v+y) = (x+v) + y = 0_V + y = y,$$

that is, $x = y$.

Q2 (15 p.) Let $V = \mathbb{R}_+ = \{v \in \mathbb{R} : v > 0\}$ be the real vector space with its vector operations $v + w = vw$ and $\lambda v = v^\lambda$, $\lambda \in \mathbb{R}$. Show that $\mathrm{Span}\{v\} = V$ for each $v \in V$, $v \neq 0_V$. Find $\dim(V) = ?$ Note that $0_V = 1$, and $v \neq 1$.

Take $w \in V$. Then $w = \lambda \cdot v$ with $\lambda = \frac{\ln(w)}{\ln(v)}$.

Indeed,

$$w = e^{\ln(w)} = e^{\frac{\ln(w)}{\ln(v)}\ln(v)} = \left(e^{\ln(v)}\right)^{\frac{\ln(w)}{\ln(v)}} = v^{\lambda} = \lambda \cdot v$$

In particular, $\dim(V) = 1$.

Q3 (25 p.) Let $S = \{a, b, c, d\}$ be a set with its four different points, and let $V = \{f \in \mathrm{Fun}(S) : 2f(a) + f(d) = 0, 3f(b) - f(c) = 0\}$. Find a basis B for V, and $\dim(V) = ?$.

Take $f \in V$. Then $f = f(a)\chi_a + f(b)\chi_b + f(c)\chi_c + f(d)\chi_d$.

But $f(d) = -2f(a)$ and $f(c) = 2f(b)$. Therefore

$$f = f(a)(\chi_a - 2\chi_d) + f(b)(\chi_b + 2\chi_c).$$

Note that $\chi_a - 2\chi_d, \chi_b + 2\chi_c \in V$ and

$V = \mathrm{Span}\{\chi_a - 2\chi_d, \chi_b + 2\chi_c\}$.

Actually theese vectors are linearly independent: $\lambda(\chi_a + 2\chi_d) + \mu(\chi_b + 2\chi_c) = 0$

$t = a \Rightarrow \lambda \chi_a(a) = 0 \Rightarrow \lambda = 0$

$t = b \Rightarrow \mu \chi_b(b) = 0 \Rightarrow \mu = 0$.

So, $B = \{\chi_a + 2\chi_d, \chi_b + 2\chi_c\}$ is a basis for V and $\dim(V) = |B| = 2$.

Q4 (25 p.) Consider the following vectors $\mathbf{a}_1 = (1,-3,0,1)$, $\mathbf{a}_2 = (2,0,-1,1)$, $\mathbf{a}_3 = (4,-6,-1,3)$, $\mathbf{a}_4 = (3,-9,0,3)$ from the vector space $\mathbb{R}^4$. Find $\dim(\mathrm{Span}\,\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}) = ?$. Explain your answer.

First note that $\vec{a}_4 = 3\vec{a}_1$. Further $\lambda\vec{a}_1 + \mu\vec{a}_2 + \theta\vec{a}_3 = \vec{0}$ implies that

$$\begin{cases} \lambda + 2\mu + 4\theta = 0 \\ -3\lambda \qquad - 6\theta = 0 \Rightarrow \theta = -\frac{\lambda}{2} \\ \qquad -\mu - \theta = 0 \Rightarrow \theta = -\mu \\ \lambda + \mu + 3\theta = 0 \end{cases}$$

Whence $\lambda = 2\mu$, $\theta = -\mu$. But λ is free.

In particular, $\lambda = 2$, $\mu = 1$, $\theta = -1$ is a nontrivial solution to the linear system.

Thus $2\vec{a}_1 + \vec{a}_2 = \vec{a}_3$. But obviously, $\vec{a}_1$ and $\vec{a}_2$ are linearly independent vectors.

Consequently,

$\mathrm{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$ and

$\dim(\mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}) = 2$.

Q5 (25 p.) Let $\mathbb{C}^3$ be three-dimensional complex vector space. Consider its subspace $V = \{(x, y, z) \in \mathbb{C}^3 : ix + (1-i)\,y - 5z = 0\}$. Find its basis, and $\dim_{\mathbb{C}}(V) =?$. Explain what does V geometrically mean?

Choose the vectors from V:

$\vec{f}_1 = (-5i,\ 0,\ 1)$, $\vec{f}_2 = (-1-i,\ -1,\ 0)$

They are linearly independent over $\mathbb{C}$.

Take $(x, y, z) \in V$. Then

$(x, y, z) = \lambda \vec{f}_1 + \mu \vec{f}_2$ means that

$$\begin{cases} -5i\lambda - (1+i)\mu = x \\ -\mu = y \\ \lambda = z \end{cases}$$

Thus $\lambda = z$, $\mu = -y$ and

$$-5iz + (1+i)y = i\left(-5z + \frac{1+i}{i}y\right) =$$

$$= i\left((1-i)y - 5z\right) = i(-ix) = x$$

Hence $V = \operatorname{Span}\{\vec{f}_1, \vec{f}_2\}$ and $\dim_{\mathbb{C}}(V) = 2$.

Geometrically V is a plane over $\mathbb{C}$, or just a complex plane.