M E T U
Northern Cyprus Campus

Math 219	Differential Equations	II Short Exam	19.03.2015
Last Name : Name : Student No:	Dept./Sec. : Time : 11: 40 Duration : 40 *minutes*		Signature
2 QUESTIONS			TOTAL 10 POINTS
1	2		

Q1 (5 pts.) Find the fundamental matrix $\Phi(t)$ (recall that $\Phi(t) = \Psi(t)\,P^{-1}$ or $\Psi(t)\,T^{-1}$) of the system $\mathbf{x}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}(t)$. Sketch the phase portrait of solutions.

$$\Delta(t) = \begin{vmatrix} 1-t & -4 \\ 4 & -7-t \end{vmatrix} = (t+3)^2 = 0 \Rightarrow \sigma(A) = \{-3^{(2)}\}$$

$$A + 3I = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow V_{-3} = \{x = y\}.$$

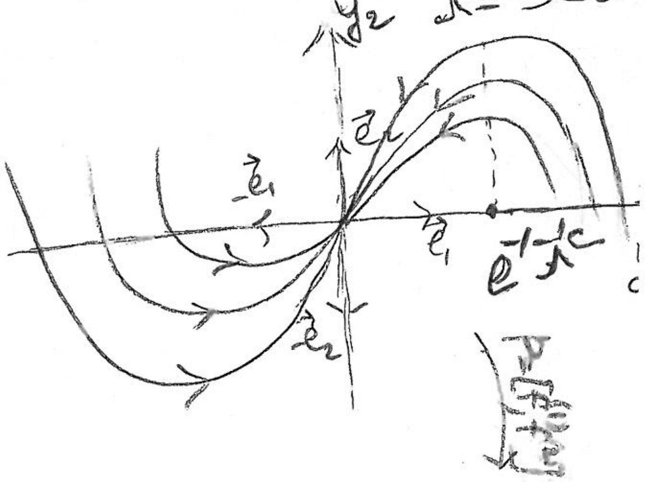
So, $\dim(V_{-3}) = 1 < 2 = \text{alg. mul}(-3)$

Put $\vec{f}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{f}^{(2)} = (A+3I)\vec{f}^{(1)} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

Then $P = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/4 \end{bmatrix}$

$J = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}$, $\Psi(t) = Pe^{Jt} =$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ te^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + 4te^{-3t} & 4e^{-3t} \\ 4te^{-3t} & 4e^{-3t} \end{bmatrix}$$

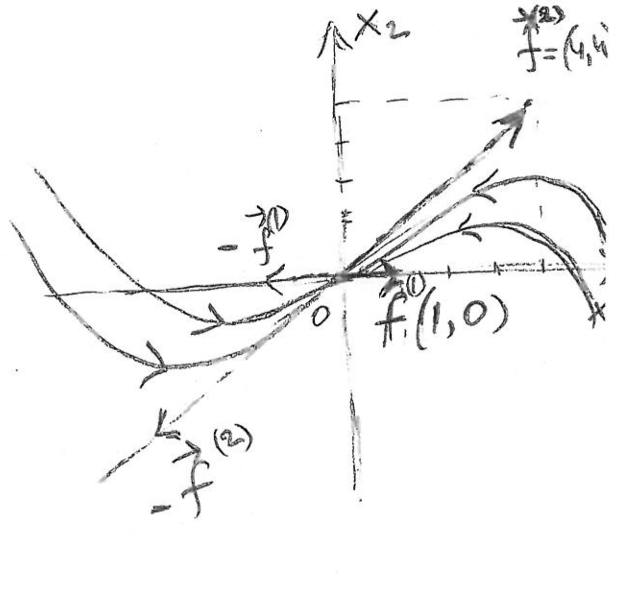


$\Phi(t) = \Psi(t) \cdot P^{-1}$

$$\Phi(t) = \begin{bmatrix} e^{-3} + 4te^{-3t} & 4e^{-3t} \\ 4te^{-3t} & 4e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1/4 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & -4te^{-3t} + e^{-3t} \end{bmatrix}$$

$\Phi(0) = I$



Q2

$$\det(A-tI)=\begin{vmatrix}-t & 1 & 0\\ 0 & -t & 1\\ 8 & -12 & 6-t\end{vmatrix}=t^3-6t^2+12t-8=0$$

$t=2$ (3)

$\sigma(A)=\{2^{(3)}\}$

Q2 (5 pts.) Find the fundamental matrix $\Psi(t)$ of the system $\mathbf{x}'(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 8 & -12 & 6\end{bmatrix}\mathbf{x}(t)$.

$$\lambda=2 \Rightarrow A-2=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\sim\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow$$

$\Rightarrow 2x-y=0$ and $2y-z=0$ a line $(1,2,4)$ $\Rightarrow m(2)=1<3=alg(2)$

Compute $V_{2,2}=\ker(A-2)^2$

$$\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}=\begin{bmatrix}4 & -4 & 1\\ 8 & -8 & 2\\ 16 & -16 & 4\end{bmatrix}\sim\begin{bmatrix}4 & -4 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\Rightarrow 4x-4y+z=0 \text{ a plane.}$$

So, $V_{2,1}=\operatorname{span}(1,2,4)\subsetneq V_{2,2}=\{z=4y-4x\}\leq V_{2,3}=\mathbb{C}^3\Rightarrow \overline{V_2}=\mathbb{C}^3$

put $\vec{f}^{(1)}=(0,0,1)\notin V_{2,2}\Rightarrow f^{(2)}=(A-2)\vec{f}^{(1)}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 4\end{bmatrix}\in V_{2,2}$

$\vec{f}^{(3)}=(A-2)\vec{f}^{(2)}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\begin{bmatrix}0\\ 1\\ 4\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}\in V_{2,1}$

So, $f^{(1)}=(0,0,1)$, $f^{(2)}=(0,1,4)$, $f^{(3)}=(1,2,4)$

$$P=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 2\\ 1 & 4 & 4\end{bmatrix},\ J=\begin{bmatrix}2 & 0 & 0\\ 1 & 2 & 0\\ 0 & 1 & 2\end{bmatrix}\Rightarrow \Psi(t)=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 2\\ 1 & 4 & 4\end{bmatrix}\begin{bmatrix}e^{2t} & 0 & 0\\ te^{2t} & e^{2t} & 0\\ \frac{t^2}{2}e^{2t} & te^{2t} & e^{2t}\end{bmatrix}=$$

$$=\begin{bmatrix}\frac{t^2}{2}e^{2t} & te^{2t} & e^{2t}\\ te^{2t}+t^2e^{2t} & e^{2t}+2te^{2t} & 2e^{2t}\\ e^{2t}+4te^{2t}+2t^2e^{2t} & 4e^{2t}+4te^{2t} & 4e^{2t}\end{bmatrix}$$

If put $\vec{f}^{(1)}=(1,0,0)\notin V_{2,2}\Rightarrow \vec{f}^{(2)}=(A-2)\vec{f}^{(1)}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}-2\\ 0\\ 8\end{bmatrix},\ f^{(3)}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 8 & -12 & 4\end{bmatrix}\begin{bmatrix}-2\\ 0\\ 8\end{bmatrix}$

$$=\begin{bmatrix}4\\ 8\\ 16\end{bmatrix}\Rightarrow P=\begin{bmatrix}1 & -2 & 4\\ 0 & 0 & 8\\ 0 & 8 & 16\end{bmatrix}\quad \Psi(t)=\begin{bmatrix}1 & -2 & 4\\ 0 & 0 & 8\\ 0 & 8 & 16\end{bmatrix}\begin{bmatrix}e^{2t} & 0 & 0\\ te^{2t} & e^{2t} & 0\\ \frac{t^2}{2}e^{2t} & te^{2t} & e^{2t}\end{bmatrix}=$$

$$=\begin{bmatrix}e^{2t}-2te^{2t}+2t^2e^{2t} & -2e^{2t}+4te^{2t} & 4e^{2t}\\ 4t^2e^{2t} & 8te^{2t} & 8e^{2t}\\ 8te^{2t}+8t^2e^{2t} & 8e^{2t}+16te^{2t} & 16e^{2t}\end{bmatrix}$$