M E T U
Northern Cyprus Campus

Math 219	Differential Equations	Midterm Exam	05.04.2015
Last Name : Name : KEY Student No :	Dept./Sec.: Time : 09: 40 Duration : 120 *minutes*	Signature	

5 QUESTIONS ON 4 PAGES									TOTAL 100 POINTS
1	2	3	4	5					

Q1 (20 p.) Find the solution to the IVP $\begin{cases} \dfrac{dy}{dx} = \dfrac{x+3y}{x-y}, \\ y(1) = -1. \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{x}{x}+3\frac{y}{x}}{\frac{x}{x}-\frac{y}{x}} = \frac{1+3\frac{y}{x}}{1-\frac{y}{x}} \Rightarrow v + x\frac{dv}{dx} = \frac{1+3v}{1-v}$$

$$\boxed{\begin{aligned} v = \frac{y}{x} \Rightarrow y = vx \\ \frac{dy}{dx} = v + x\frac{dv}{dx} \end{aligned}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v-v(1-v)}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2+2v+1}{1-v} \quad \text{(Seperable)}$$

Then, we get $\displaystyle\int \frac{v-1}{(v+1)^2}\,dv + \int \frac{1}{x}\,dx = C \quad (v \neq -1)$

$$\int \frac{(v+1)-2}{(v+1)^2}\,dv + \int \frac{1}{x}\,dx = C \Rightarrow \int \frac{1}{v+1}\,dv - 2\int \frac{1}{(v+1)^2}\,dv + \int \frac{1}{x}\,dx = C$$

$$\ln|v+1| + 2\frac{1}{(v+1)} + \ln|x| = C \underset{\text{put } v = \frac{y}{x}}{\Rightarrow} \ln\left|\frac{y}{x}+1\right| + 2\frac{1}{\frac{y}{x}+1} + \ln|x| = C$$

$$\therefore \ln\left|\frac{y+x}{x}\right| + 2\frac{x}{y+x} + \ln|x| = C \Rightarrow \ln|y+x| + 2\frac{x}{y+x} = C$$

Need to check $v = -1$: $\frac{y}{x} = -1 \Rightarrow y = -x$

$-1 = \frac{dy}{dx} = \frac{x+3\cdot(-x)}{x-(-x)} = -1$ ✓ $y = -x$ is also a solution.

Actually, $y = -x$ is the solution with $y(1) = -1$.

Q2 (25 p.) Consider the linear system $\mathbf{x}'(t) = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}(t)$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Find the **REAL** fundamental matrix $\Psi(t)$ to the system and solve IVP. **Bonus (5 p.):** Sketch the phase curve of the solution to IVP.

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

$$\sigma(A) = \{-1+i, -1-i\}$$

$\underline{\lambda = -1+i}$ $\quad (A-(-1+i)I)\vec{\xi} = \vec{0}$

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \xrightarrow{-(2+i)R_1 + R_2} \begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{\xi} = k \cdot \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$X^{(1)}(t) = e^{(-1+i)t} \cdot \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \cdot (\cos(t) + i\sin(t)) \cdot \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin t \\ -\cos t + 2\sin t \end{bmatrix}$$

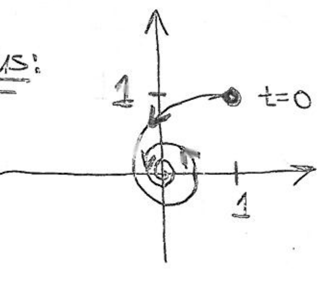
$$\Psi(t) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ 2e^{-t}\cos t + e^{-t}\sin t & -e^{-t}\cos t + 2e^{-t}\sin t \end{bmatrix} \Rightarrow \det \Psi(t) = -e^{-2t} \neq 0$$

<u>I.V.P.</u>

$$\Psi(0) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{array}{r} c_1 = 1 \\ 2c_1 - c_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}$$

$$\therefore X(t) = 1 \cdot \begin{bmatrix} e^{-t}\cos t \\ 2e^{-t}\cos t + e^{-t}\sin t \end{bmatrix} + 1 \cdot \begin{bmatrix} e^{-t}\sin t \\ -e^{-t}\cos t + 2e^{-t}\sin t \end{bmatrix}$$

<u>Bonus:</u>



Q3 (25 pts) Find the general solution to the linear system $\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \mathbf{x}(t)$.

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = (2-\lambda)\left((2-\lambda)^2-1\right) + 1\cdot(-2+\lambda-1) + (-1-2+\lambda)$$

$$= 8-12\lambda+6\lambda^2-\lambda^3-2+\lambda-2+\lambda-1-1-2+\lambda$$

$$= -\lambda^3+6\lambda^2-9\lambda = -\lambda(\lambda^2-6\lambda+9) \Rightarrow \sigma(A)=\{3^{(2)}, 0^{(1)}\}$$

$\underline{\lambda=3}$ $\quad (A-3I)\vec{\xi}=\vec{0}$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_1+R_3]{-R_1+R_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} \xi_1 &= -\xi_2+\xi_3 \\ \xi_2 &= \xi_2 \\ \xi_3 &= \xi_3 \end{aligned} \qquad V_3 = \left\{ k\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + l\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda=0}$ $\quad (A-0\cdot I)\vec{\xi}=\vec{0}$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[\substack{R_1+R_2 \\ -2R_1+R_3}]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow[\frac{1}{3}R_2]{R_2+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} \xi_1 &= -\xi_2-2\xi_3 = -\xi_3 \\ \xi_2 &= -\xi_3 \\ \xi_3 &= \xi_3 \end{aligned}$$

$$V_0 = \left\{ k\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\vec{X}_g(t) = c_1 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Q4 (10 pts) Show that if y_1 and y_2 are solutions to $y'' + p(t)\,y' + q(t)\,y = r(t)$ then $y_1 - y_2$ is a solution to $y'' + p(t)\,y' + q(t)\,y = 0$.

$$\begin{aligned} & y_1'' + p(t)\,y_1' + q(t)\cdot y_1 = r(t) \\ - \quad & y_2'' + p(t)\cdot y_2' + q(t)\cdot y_2 = r(t) \\ \hline & y_1''-y_2'' + p(t)\cdot(y_1'-y_2') + q(t)\cdot(y_1-y_2) = 0 \end{aligned}$$

that's

$(y_1-y_2)'' + p(t)\cdot(y_1-y_2)' + q(t)\cdot(y_1-y_2) = 0$ i.e. y_1-y_2 is a solution to the homogeneous eqn. ∎

Q5 (7+6+7=20 pts) Sketch the phase portraits of 2×2-linear systems $\mathbf{x}'(t) = A\mathbf{x}(t)$ whose P and J matrices (recall that $A = PJP^{-1}$) are given below.

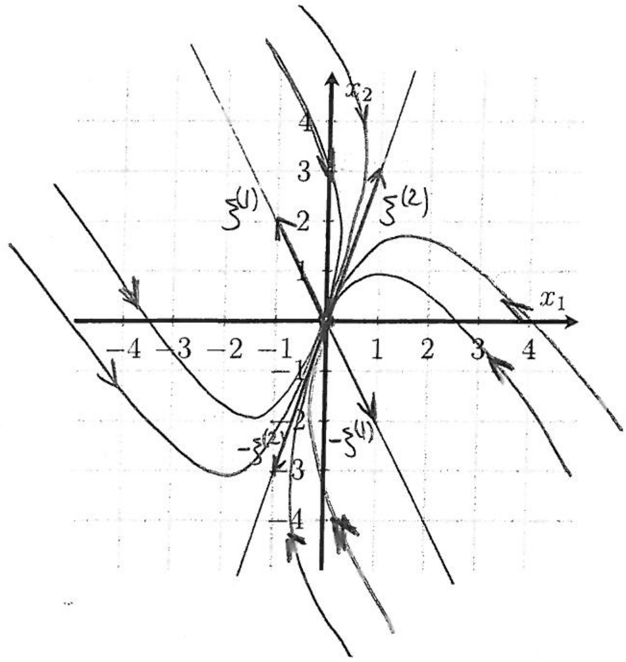
(a) $P = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, $J = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}$;

$\lambda_1 = -5, \quad \lambda_2 = -3$

$\xi^{(1)} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$y_1(t) = c_1 e^{-5t} \quad y_2(t) = c_2 e^{-3t}$

$y_1 = C \cdot y_2^{-5/3} = C \cdot y_2^{5/3}$



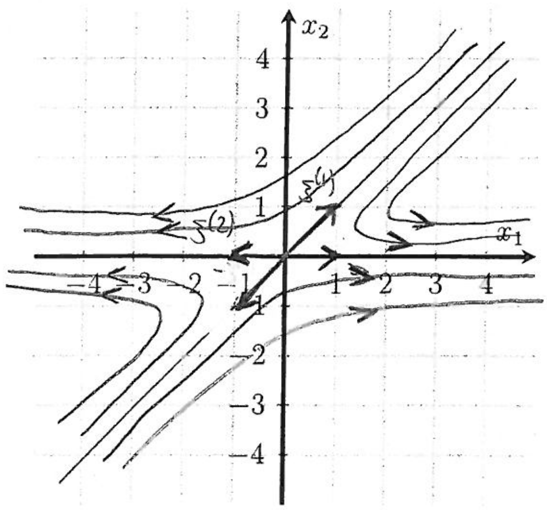
(b) $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $J = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$;

$\lambda_1 = -2, \quad \lambda_2 = 1$

$\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \xi^{(2)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$y_1(t) = c_1 e^{-2t} \quad y_2(t) = c_2 e^{t}$

$y_2 = C \cdot y_1^{-1/2} = \dfrac{C}{y^{1/2}}$



(c) $P = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$;

$\lambda_1 = 0 \quad \lambda_2 = 5$

$\xi^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \xi^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1(t) = c_1 \quad y_2(t) = c_2 e^{5t}$