M E T U
Northern Cyprus Campus

	Introduction to Differential Equations Final Exam	
Code : *Math 219* Acad.Year: *2013-2014* Semester : *Fall* Date : *21.01.2014* Time : *09:00* Duration : *150 minutes*	Last Name: Name: KEY Department: Signature:	Student No: Section:
	6 QUESTIONS ON 6 PAGES TOTAL 105 POINTS	

1 (20)	2 (16)	3 (20)	4 (16)	5 (17)	6 (16)	

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1.(20 pts) Determine the general solution of the differential equation.

$$y''' + y'' + y' + y = e^{-t} + 4t$$

$y''' + y'' + y' + y = 0$ has characteristic polynomial $r^3 + r^2 + r + 1$

$r^3 + r^2 + r + 1 = r^2(r+1) + (r+1) = (r^2+1)(r+1) \Rightarrow$ roots are $\mp i, -1$.

Hence, $y_h(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t$

By the method of undetermined coefficients $y_p(t) = A \cdot t \cdot e^{-t} + (Bt + C)$

$y_p'(t) = Ae^{-t} - Ate^{-t} + B$

$y_p''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + At \cdot e^{-t}$

$y_p'''(t) = 2Ae^{-t} + Ae^{-t} - Ate^{-t} = 3Ae^{-t} - Ate^{-t}$

By putting them into the equation.

$3Ae^{-t} - Ate^{-t} - 2Ae^{-t} + Ate^{-t} + Ae^{-t} - Ate^{-t} + B + Ate^{-t} + Bt + C = e^{-t} + 4t$

$(3A - 2A + A - \cancel{At} + \cancel{At} - \cancel{At} + \cancel{At})e^{-t} + B \cdot t + (C + B) = e^{-t} + 4t$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$B = 4 \Rightarrow B = 4$

$C + B = 0 \Rightarrow C = -4$

$\therefore\ y_g(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t + \frac{1}{2} t e^{-t} + 4t - 4$

2.(16 pts) Consider the system shown in the figure, where an object of mass m is attached to a linear spring with spring constant k. Assume that there is no friction or damping. An external force $F(t)$ is applied to the mass m. The system is initially at rest.

(a) Write the equation of motion for the object by considering all forces on the object.

$y(t) \equiv$ position of the mass from equilibrium

$$my'' + 0\cdot y' + k\cdot y = F(t) \quad y(0)=0,\ y'(0)=0$$

$$my'' + k\cdot y = F(t) \quad y(0)=0,\ y'(0)=0$$

(b) Convert the equation of motion to a 2×2 linear system.

$x_1 = y,\ x_2 = y' \Rightarrow x_1' = y' = x_2$

$$x_2' = y'' = -\frac{k}{m}y + \frac{F(t)}{m} = -\frac{k}{m}x_1 + \frac{F(t)}{m}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ +\frac{F(t)}{m} \end{bmatrix}$$

(c) Suppose that $k = m = 1$ and $F(t) = \sin t$. Find the solution of the system using variation of parameters.

$$y'' + y = \sin t \quad y(0)=0,\ y'(0)=0$$

Ch. Poly: $r^2+1=0 \Rightarrow r = \mp i \quad y_h(t) = c_1\cos t + c_2 \sin t.$

Then, the corresponding system has fundamental matrix $\Psi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

By variation of parameters, $y_p(t) = c_1(t)\cdot\cos t + c_2(t)\cdot\sin t$ has equation of the form

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin t \end{bmatrix} \Rightarrow \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \frac{1}{1}\cdot\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ \sin t \end{bmatrix} = \begin{bmatrix} -\sin^2 t \\ \cos t\cdot\sin t \end{bmatrix}$$

$$c_1(t) = \int -\sin^2 t\, dt = \int \frac{(\cos 2t - 1)}{2}\, dt = +\frac{\sin 2t}{4} - \frac{t}{2} + c_1$$

$$c_2(t) = \int \cos t \sin t\, dt = \int \frac{\sin(2t)}{2}\, dt = -\frac{\cos(2t)}{4} + c_2$$

$$y_g(t) = c_1\cos t + c_2\sin t - \frac{t}{2}\cos t + \frac{\sin 2t\cdot\cos t}{4} - \frac{\cos(2t)\sin t}{4} = \frac{\sin t}{4}$$

$0 = y_g(0) = c_1$

$0 = y_g'(0) = c_2\cos(0) - \frac{\cos(0)}{2} + \frac{\cos(0)}{4} \Rightarrow c_2 = 1/4$

3.(20 pts) Find the solution to the initial value problem

$$y'' - 4y = \delta(t-1) + f(t) \quad y(0) = 0, \; y'(0) = 3 \quad \text{where } f(t) = \begin{cases} 2t & t < 2 \\ 6-t & 2 \leq t \end{cases}$$

$$f(t) = 2t + u_2(t)\cdot(6-3t) = 2t - 3u_2(t)(t-2)$$

By using Laplace Transform

$$s^2 \cdot Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{3} - 4Y(s) = e^{-s} + \frac{2}{s^2} + e^{-2s}\cdot\frac{3}{s^2}$$

$$Y(s)(s^2-4) = e^{-s} + \frac{2}{s^2} + e^{-2s}\cdot\frac{3}{s^2} + 3$$

$$Y(s) = e^{-s}\cdot\frac{1}{s^2-4} + \frac{2}{s^2(s^2-4)} + e^{-2s}\cdot\frac{3}{s^2(s^2-4)} + \frac{3}{s^2-4}$$

$$\frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} \Rightarrow A = \frac{1}{4}, \; B = -\frac{1}{4}$$

$$\frac{1}{s^2(s^2-4)} = \frac{As+B}{\underset{(s^2-4)}{s^2}} + \frac{C}{\underset{s^2(s+2)}{s-2}} + \frac{D}{\underset{s^2(s-2)}{s+2}} = \frac{As^3 - 4As + Bs^2 - 4B + Cs^3 + 2Cs^2 + Ds^3 - 2Ds^2}{s^2(s^2-4)}$$

$$\left.\begin{aligned} A + C + D &= 0 \\ B + 2C - 2D &= 0 \end{aligned}\right\} \Rightarrow 2A + B + 4C = 0 \Rightarrow C = \frac{1}{16} \Rightarrow D = -\frac{1}{16}$$

$$-4A = 0 \Rightarrow A = 0$$

$$-4B = 1 \Rightarrow B = -\frac{1}{4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\Big\{ e^{-s}\left(\frac{\frac{1}{4}}{s-2} + \frac{-\frac{1}{4}}{s+2}\right) + 2\left(\frac{-\frac{1}{4}}{s^2} + \frac{\frac{1}{16}}{s-2} + \frac{-\frac{1}{16}}{s+2}\right)$$

$$+ e^{-2s}\cdot 3\left(\frac{-\frac{1}{4}}{s^2} + \frac{\frac{1}{16}}{s-2} + \frac{-\frac{1}{16}}{s+2}\right) + 3\cdot\left(\frac{\frac{1}{4}}{s-2} + \frac{-\frac{1}{4}}{s+2}\right)\Big\}$$

$$= u_1(t)\left(\frac{1}{4}e^{+2(t-1)} - \frac{1}{4}e^{-2(t-1)}\right) - \frac{1}{2}\cdot t + \frac{1}{8}e^{2t} - \frac{1}{8}e^{-2t} + 3u_2(t)\left(\frac{-1}{4}(t-2) + \frac{1}{16}e^{2(t-2)} - \frac{1}{16}e^{-2(t-2)}\right)$$

$$+ 3\left(\frac{1}{4}e^{+2t} - \frac{1}{4}e^{-2t}\right)$$

4.(4x4=16 pts) This problem has four unrelated parts.

(a) By using definition of Laplace transform compute $\mathscr{L}\{t+3\}$. Using results from the table directly will result in ZERO POINTS.

$$\mathcal{L}\{t+3\} = \int_0^\infty e^{-st}(t+3)\,dt = \lim_{A\to\infty} \int_0^A e^{-st}\cdot t\,dt + 3\cdot\int_0^A e^{-st}\,dt$$

$u = t \quad du = dt$

$dv = e^{-st}dt \quad v = \frac{e^{-st}}{-s}$

$$= \lim_{A\to\infty}\left(-\frac{te^{-st}}{s} + \int_0^A \frac{e^{-st}}{s}dt + 3\cdot\int_0^A e^{-st}dt\right) = \lim_{A\to\infty}\left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} - 3\frac{e^{-st}}{s}\right)\Bigg|_0^A$$

$$= \lim_{A\to\infty}\left(\underbrace{\frac{-A\cdot e^{-sA}}{s}}_{\to 0} \underbrace{-\frac{e^{-sA}}{s^2}}_{\to 0} \underbrace{-3\frac{e^{-sA}}{s}}_{\to 0} + 0 + \frac{1}{s^2} + 3\cdot\frac{1}{s}\right) = \frac{1}{s^2} + \frac{3}{s}$$

(b) Compute inverse Laplace transform of $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$

$$F(s) = 2\cdot e^{-2s}\cdot\frac{(s-1)}{(s-1)^2+1^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2\,u_2(t)\cdot e^{(t-2)}\cdot\cos(t-2)$$

(c) By using the definition of convolution, compute $2 * \cos t$ at $t = 5$.

$$(2*\cos t)\Big|_{t=5} = \int_0^5 2\cdot\cos(\tau)\,d\tau = 2\cdot\sin(\tau)\Big|_0^5 = 2\sin(5)$$

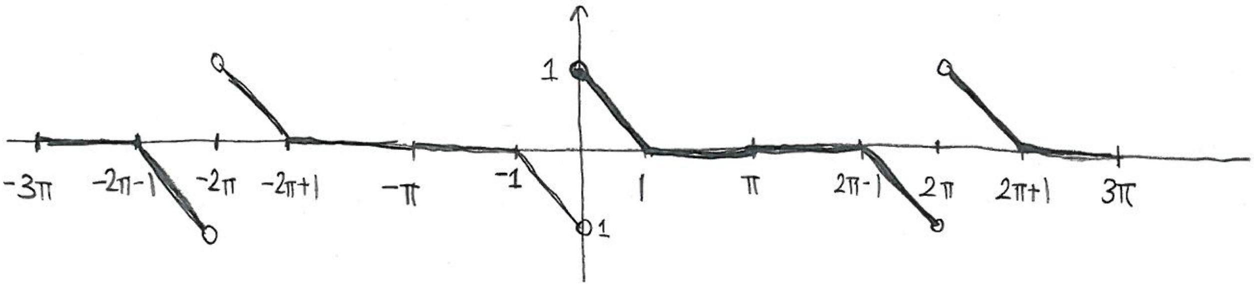
(d) Show that if $(f * f)(t) = 0$ for all t, then $f(t) = 0$

$$0 = \mathcal{L}\{f*f\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{f\} = (\mathcal{L}\{f\})^2 \Rightarrow \mathcal{L}\{f\} = 0$$

$\mathcal{L}^{-1}\{0\} = 0$ for all t.

5.(4+9+4=17 pts) Let $f(x) = 1 - x$, $0 \le x \le 1$ and $f(x) = 0$, $1 \le x \le \pi$ be a function defined on the interval $[0, \pi]$

(a) Extend it to the interval $[-\pi, \pi]$ as an odd function, and then to the real line $(-\infty, +\infty)$ as a periodic function of period 2π. Sketch the graph of the resulting function.



b) Find the sine Fourier series $S_f(x)$ of the function $f(x)$ and calculate $S_f(-7)$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\cdot\sin\left(\frac{n\pi}{\pi}x\right)dx = 2\cdot\frac{1}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx = \frac{2}{\pi}\int_0^1 (1-x)\sin(nx)\,dx$$

$u = 1-x \quad du = -dx$

$dv = \sin(nx) \quad v = -\frac{\cos(nx)}{n}$

$$= \frac{2}{\pi}\left(-\frac{(1-x)\cdot\cos(nx)}{n} - \int_0^1 \frac{\cos(nx)}{n}\,dx\right)$$

$$= \frac{2}{\pi}\left(\frac{(x-1)\cdot\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right)\Bigg|_0^1 = \frac{2}{\pi}\left(0 - \frac{\sin(n)}{n^2} + \frac{1}{n} + 0\right)$$

$$= \frac{2}{n\pi}\left(1 - \frac{\sin(n)}{n}\right)$$

$$S_f(-7) = S_f(-7+2\pi) \underset{\uparrow}{=} \tilde{f}(-7+2\pi) = -(-7+2\pi) - 1 = 6 - 2\pi.$$

$\tilde{f}(x)$ is cont.
at $-7+2\pi$
since $-1 < -7+2\pi < 0$

c) Solve the following heat conduction problem

$$u_{xx} = u_t,\; u(0,t) = u(\pi,t) = 0, t > 0,\; u(x,0) = f(x), 0 < x < \pi,$$

where $f(x)$ is the function from item a).

$\alpha^2 = 1, \quad L = \pi$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-1\cdot\frac{n^2\cancel{\pi^2}}{\cancel{\pi^2}}t}\underbrace{\frac{2}{n\pi}\left(1 - \frac{\sin(n)}{n}\right)}_{b_n}\cdot\sin(nx)$$

6.(16 pts) Consider the partial differential equation and the boundary conditions below

$$u_{xx} = 4u_{yy}, \quad u(0, y) = u(2, y) = 0.$$

where $u(x, y)$ is a function of 2 variables. Find all nontrivial solutions of this problem of the form $u(x, y) = X(x)Y(y)$.

a) $u(x,y) = X(x)\cdot Y(y) \Rightarrow \dfrac{X''(x)\cdot Y(y)}{Y(y)\cdot X(x)} = 4\dfrac{X(x)\cdot Y''(y)}{Y(y)\cdot X(x)}$, then we get

$$\frac{X''(x)}{X(x)} = \frac{4Y''(y)}{Y(y)} = -\lambda \Rightarrow \begin{array}{l} X''(x) + \lambda X(x) = 0 \\ 4Y''(y) + \lambda Y(y) = 0 \end{array}$$

$0 = u(0,y) = X(0)\cdot Y(y) \Rightarrow X(0) = 0$

$0 = u(2,y) = X(2)\cdot Y(y) \Rightarrow X(2) = 0$ OR $Y(y) = 0$ which implies $u(x,y) = 0$.

b) $X'' + \lambda X = 0 \quad X(0) = X(2) = 0 \quad$ Ch. Poly. $r^2 + \lambda = 0$

Case 1: $\lambda < 0 \quad X(x) = c_1 e^{+\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ $\quad \left.\begin{array}{l} 0 = X(0) = c_1 + c_2 \\ 0 = X(2) = c_1 e^{\sqrt{\lambda}\cdot 2} + c_2 e^{-\sqrt{\lambda}\cdot 2} \end{array}\right\} \Rightarrow c_1 = c_2 = 0$

Case 2: $\lambda = 0 \quad X(x) = c_1 + c_1 x$ $\quad \left.\begin{array}{l} 0 = X(0) = c_1 \\ 0 = X(2) = c_1 + 2c_2 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$

Case 3: $\lambda > 0 \quad X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ $\quad \left.\begin{array}{l} 0 = X(0) = c_1 \\ 0 = X(2) = c_2\sin(\sqrt{\lambda}\cdot 2) \end{array}\right\} \begin{array}{l} c_1 = c_2 = 0 \\ \text{OR} \\ c_1 = 0,\ \sqrt{\lambda}\cdot 2 = n\pi \end{array}$

So, we get $\sqrt{\lambda} = \dfrac{n\pi}{2}$, and, $X_n(x) = \sin\left(\dfrac{n\pi}{2}x\right)$ are eigenfunctions

$$Y'' + \frac{\lambda}{4}Y = 0 \Rightarrow Y'' + \frac{n^2\pi^2}{16}Y = 0 \quad \text{Ch. Poly: } r^2 + \frac{n^2\pi^2}{16} = 0$$

$$r = \mp\frac{n\pi}{4}i$$

$$Y(y) = c_1\cos\left(\frac{n\pi}{4}y\right) + c_2\sin\left(\frac{n\pi}{4}y\right)$$