M E T U

Northern Cyprus Campus

Math 219 **Differential Equations** **Final Exam** **31.04.2014**		
Last Name: Name: Student No: KEY	Dept./Sec. : Time : 09: 00 Duration : 100 *minutes*	Signature
5 QUESTIONS		TOTAL 100 POINTS

1	2	3	4	5			

Q1 (15pts.) Determine a suitable form for a special solution $Y(t)$ of the differential equation $y^{(4)} - 2y'' + y = \cos(t) + e^{2t} + e^{t}$ based on Method of Undetermined Coefficients. Do not evaluate the constants.

$$y^{(4)} - 2y'' + y = 0 \Rightarrow \lambda^4 - 2\lambda^2 + 1 = 0, \quad (\lambda^2-1)^2 = 0$$

5 $$\Rightarrow \lambda = 1^{(2)}, -1^{(2)} \Rightarrow y_h = C_1 e^{t} + C_2 t e^{t} + C_3 e^{-t} + C_4 t e^{-t}$$

5 $\cos(t) \longrightarrow Y_1(t) = A\cos(t) + B\sin(t)$

$e^{2t} \longrightarrow Y_2(t) = Ce^{2t}$

5 $e^{t} \longrightarrow Y_3(t) = D \cdot e^{t} \cdot t^2$ (duplication)

Hence

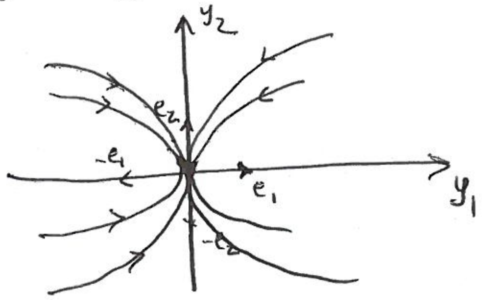
$$Y(t) = A\cos(t) + B\sin(t) + Ce^{2t} + De^{t} \cdot t^2.$$

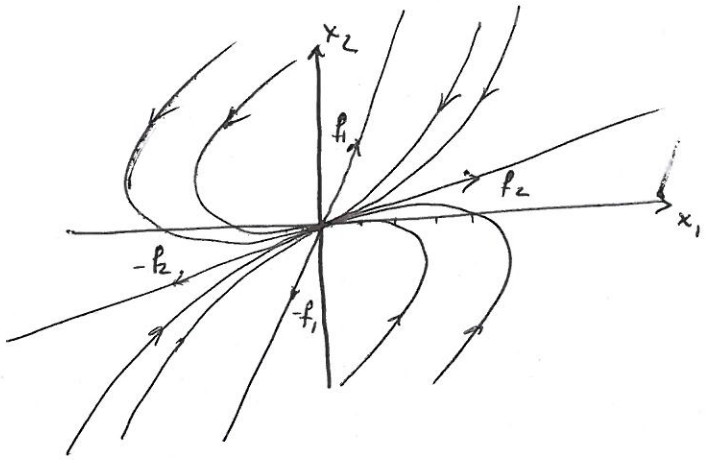
Q2 (20=5+5+10 pts.) Sketch the phase portrait of 2×2-linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with constant matrix $A \in \mathbb{M}_2(\mathbb{R})$ whose Jordan matrix J and the matrix P of (generalized) eigenvectors are given below:

a) $J = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$.

$y_2 = C y_1^{1/3}$

5

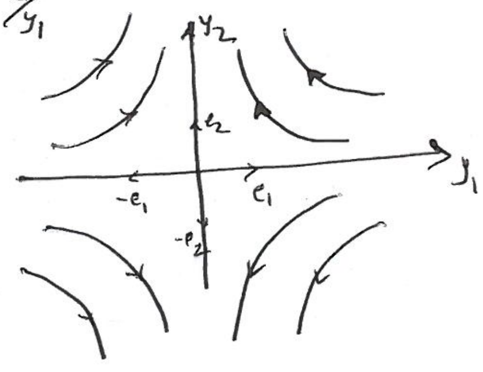


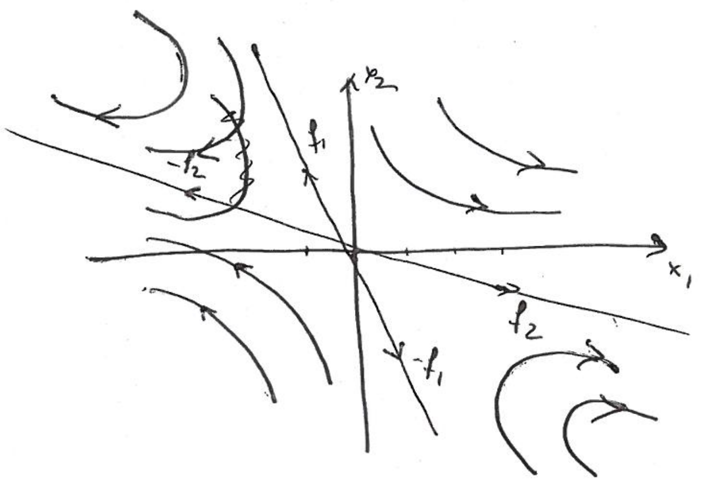


b) $J = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$.

$y_2 = C/y_1$

5

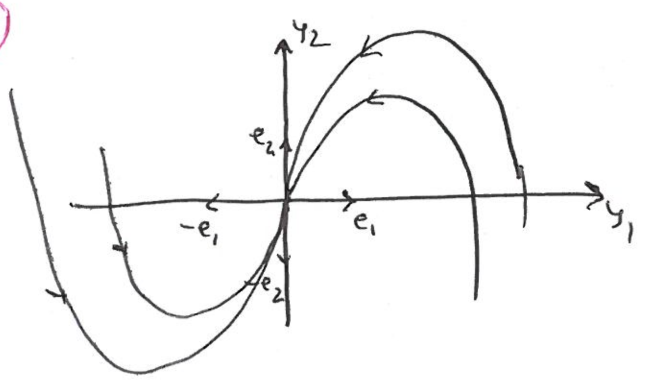


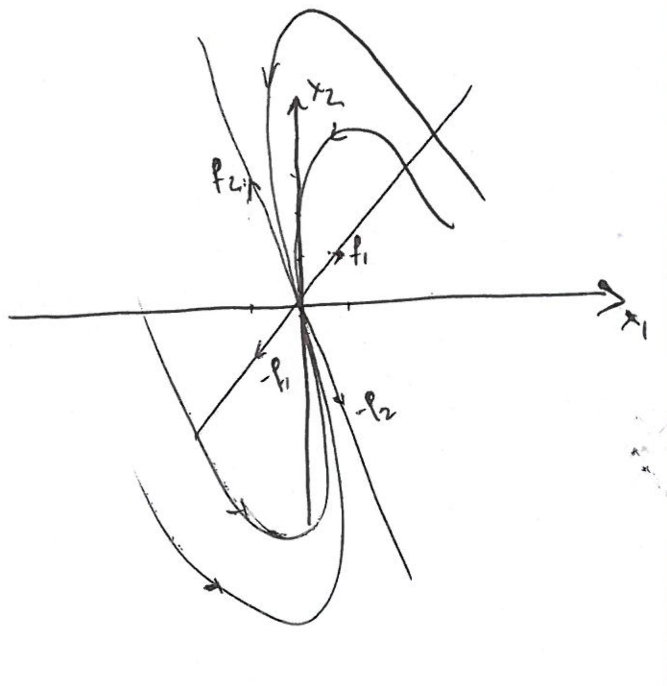


c) $J = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

$y_2 = (C - \frac{1}{3} \ln|y_1|) y_1$

10





Q3 (20 pts.) Find the solution to the given IVP: $\begin{cases} y' = \dfrac{5ty^3}{\sqrt{2+t^2}}, \\ y(0) = 1. \end{cases}$

It is a separable dif. equation:

$$\frac{dy}{y^3} = \frac{5t\,dt}{\sqrt{2+t^2}} \Rightarrow -\frac{1}{2y^2} = \frac{10}{2}(2+t^2)^{1/2} + C \Rightarrow$$

$-\dfrac{1}{y^2} = 10(2+t^2)^{1/2} + C$. But $y(0) = 1$. Therefore

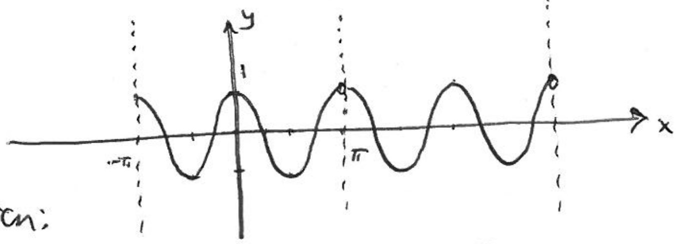
$-1 = 10\sqrt{2} + C \Rightarrow C = -1 - 10\sqrt{2}$. Thus

$$y = \frac{1}{\sqrt{10\sqrt{2} + 1 - 10\sqrt{2+t^2}}}$$

is the solution to IVP.

Q4 (20 pts.) Find the Fourier series $S_f(x)$ of the following periodic function $f(x) = \cos(2x)$, $-\pi \le x < \pi$ of period 2π. Sketch the graph of the extended function $f(x)$.

Since $f(x)$ is an even function, $S_f(x)$ is a cosine series of the function:



$$S_f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) \quad \text{with} \quad a_0 = \frac{2}{\pi}\int_0^{\pi} \cos(2x)\,dx = 0,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(2x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2x)\cos(nx)\,dx = \begin{cases} 0, & n \ne 2 \\ 1, & n = 2. \end{cases}$$

Whence $S_f(x) = \cos(2x) = f(x)$, which is supposed to be out of uniqueness of Fourier series expansion.

Q5 (25 pts.) Find the solution to IVP: $\begin{cases} y'' + 2y' + 2y = \cos(t) + \delta(t - \pi/2), \\ y(0) = y'(0) = 0. \end{cases}$ **Do not use** the convolution integral.

Let's apply Laplace transform:

$$(s^2 + 2s + 2)Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s} \Rightarrow Y(s) =$$

$$= \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}.$$ But

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$ with $\begin{cases} A = -C \\ 2A + B + D = 0 \\ 2A + 2B + C = 1 \\ 2B + D = 0 \end{cases}$

Thus $A = 1/5$, $B = 2/5$, $C = -1/5$, $D = -4/5$,

and

$$Y(s) = \frac{1}{5}\left(\frac{s+2}{s^2+1} - \frac{s+4}{(s+1)^2+1}\right) + \frac{e^{-\frac{\pi}{2}s}}{(s+1)^2+1},$$ which

in turn implies that

$$y(t) = \frac{1}{5}\left(\cos(t) + 2\sin(t) - e^{-t}\cos(t) - 3e^{-t}\sin(t)\right)$$

$$+ u_{\pi/2}(t)\, e^{-(t-\frac{\pi}{2})} \sin\left(t - \frac{\pi}{2}\right).$$