We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$- \rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \left|-\rho^2\sin\phi\right| = \rho^2\sin\phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\,dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

which is equivalent to Formula 15.9.3.

## 15.10 Exercises

**1–6** Find the Jacobian of the transformation.

1. $x = 5u - v, \quad y = u + 3v$
2. $x = uv, \quad y = u/v$
3. $x = e^{-r}\sin\theta, \quad y = e^{r}\cos\theta$
4. $x = e^{s+t}, \quad y = e^{s-t}$
5. $x = u/v, \quad y = v/w, \quad z = w/u$
6. $x = v + w^2, \quad y = w + u^2, \quad z = u + v^2$

**7–10** Find the image of the set $S$ under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; $x = 2u + 3v$, $y = u - v$
8. $S$ is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$
9. $S$ is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2$, $y = v$
10. $S$ is the disk given by $u^2 + v^2 \leqslant 1$; $x = au$, $y = bv$

**11–14** A region $R$ in the $xy$-plane is given. Find equations for a transformation $T$ that maps a rectangular region $S$ in the $uv$-plane onto $R$, where the sides of $S$ are parallel to the $u$- and $v$- axes.

11. $R$ is bounded by $y = 2x - 1$, $y = 2x + 1$, $y = 1 - x$, $y = 3 - x$
12. $R$ is the parallelogram with vertices $(0, 0)$, $(4, 3)$, $(2, 4)$, $(-2, 1)$
13. $R$ lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the first quadrant
14. $R$ is bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$, $y = 4x$ in the first quadrant

**15–20** Use the given transformation to evaluate the integral.

15. $\iint_R (x - 3y)\,dA$, where $R$ is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $x = 2u + v$, $y = u + 2v$
16. $\iint_R (4x + 8y)\,dA$, where $R$ is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$
17. $\iint_R x^2\,dA$, where $R$ is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $x = 2u$, $y = 3v$

**18.** $\iint_R (x^2 - xy + y^2)\,dA$, where $R$ is the region bounded by the ellipse $x^2 - xy + y^2 = 2$;
$x = \sqrt{2}\,u - \sqrt{2/3}\,v,\ y = \sqrt{2}\,u + \sqrt{2/3}\,v$

**19.** $\iint_R xy\,dA$, where $R$ is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $x = u/v,\ y = v$

**20.** $\iint_R y^2\,dA$, where $R$ is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$; $u = xy,\ v = xy^2$. Illustrate by using a graphing calculator or computer to draw $R$.

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**21.** (a) Evaluate $\iiint_E dV$, where $E$ is the solid enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Use the transformation $x = au,\ y = bv,\ z = cw$.

(b) The earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated by an ellipsoid with $a = b = 6378$ km and $c = 6356$ km. Use part (a) to estimate the volume of the earth.

(c) If the solid of part (a) has constant density $k$, find its moment of inertia about the $z$-axis.

**22.** An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region $R$ enclosed by two isothermal curves $xy = a$, $xy = b$ and two adiabatic curves $xy^{1.4} = c$, $xy^{1.4} = d$, where $0 < a < b$ and $0 < c < d$. Compute the work done by determining the area of $R$.

**23–27** Evaluate the integral by making an appropriate change of variables.

**23.** $\displaystyle\iint_R \frac{x - 2y}{3x - y}\,dA$, where $R$ is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

**24.** $\iint_R (x + y)e^{x^2 - y^2}\,dA$, where $R$ is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$

**25.** $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where $R$ is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, and $(0, 1)$

**26.** $\iint_R \sin(9x^2 + 4y^2)\,dA$, where $R$ is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$

**27.** $\iint_R e^{x+y}\,dA$, where $R$ is given by the inequality $|x| + |y| \leq 1$

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**28.** Let $f$ be continuous on $[0, 1]$ and let $R$ be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Show that

$$\iint_R f(x + y)\,dA = \int_0^1 uf(u)\,du$$