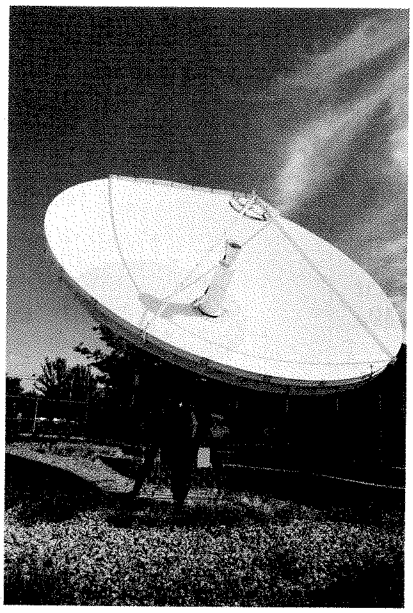
Applications of Quadric Surfaces

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 47.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 16 on page 196.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of the gears are the generating lines of the hyperboloids. See Exercise 49.)



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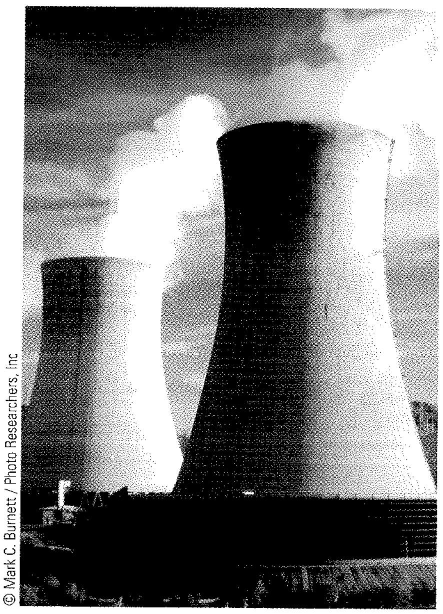
A satellite dish reflects signals to the focus of a paraboloid.



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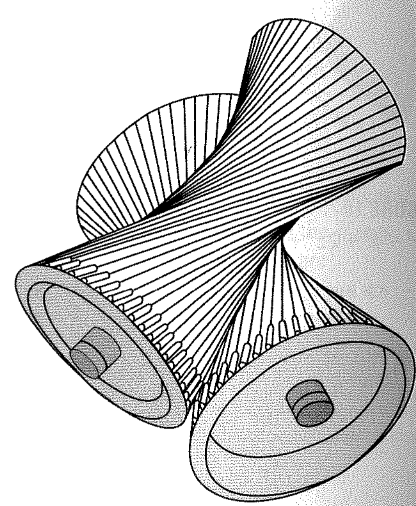
Nuclear reactors have cooling towers in the shape of hyperboloids.



Hyperboloids produce gear transmission.

12.6 Exercises

1. (a) What does the equation $y = x^2$ represent as a curve in $\mathbb{R}^2$?
   (b) What does it represent as a surface in $\mathbb{R}^3$?
   (c) What does the equation $z = y^2$ represent?

2. (a) Sketch the graph of $y = e^x$ as a curve in $\mathbb{R}^2$.
   (b) Sketch the graph of $y = e^x$ as a surface in $\mathbb{R}^3$.
   (c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

3. $y^2 + 4z^2 = 4$
4. $z = 4 - x^2$
5. $z = 1 - y^2$
6. $y = z^2$
7. $xy = 1$
8. $z = \sin y$

9. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.
   (b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?
   (c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

10. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.
(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

11–20 Use traces to sketch and identify the surface.

11. $x = y^2 + 4z^2$

12. $9x^2 - y^2 + z^2 = 0$

13. $x^2 = y^2 + 4z^2$

14. $25x^2 + 4y^2 + z^2 = 100$

15. $-x^2 + 4y^2 - z^2 = 4$

16. $4x^2 + 9y^2 + z = 0$

17. $36x^2 + y^2 + 36z^2 = 36$

18. $4x^2 - 16y^2 + z^2 = 16$

19. $y = z^2 - x^2$

20. $x = y^2 - z^2$

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

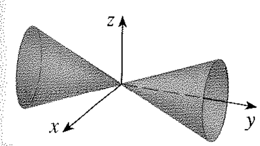
24. $-x^2 + y^2 - z^2 = 1$

25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

I


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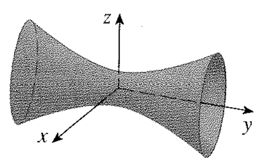

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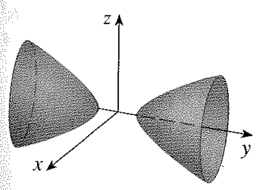

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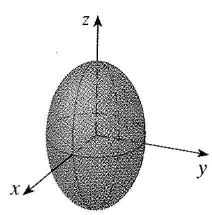

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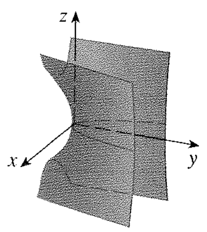

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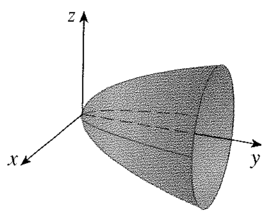

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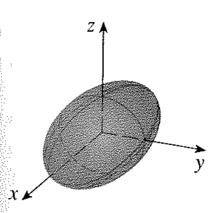
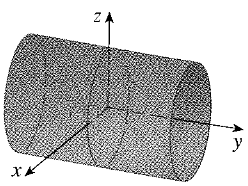


VIII


29–36 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

29. $y^2 = x^2 + \frac{1}{9}z^2$

30. $4x^2 - y + 2z^2 = 0$

31. $x^2 + 2y - 2z^2 = 0$

32. $y^2 = x^2 + 4z^2 + 4$

33. $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$

34. $4y^2 + z^2 - x - 16y - 4z + 20 = 0$

35. $x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$

36. $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$

37–40 Use a computer with three-dimensional graphing software to graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

37. $-4x^2 - y^2 + z^2 = 1$

38. $x^2 - y^2 - z = 0$

39. $-4x^2 - y^2 + z^2 = 0$

40. $x^2 - 6x + 4y^2 - z = 0$

41. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leqslant z \leqslant 2$.

42. Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

43. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the y-axis.

44. Find an equation for the surface obtained by rotating the line $x = 3y$ about the x-axis.

45. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

46. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

47. Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive z-axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.
(a) Find an equation of the earth's surface as used by WGS-84.
(b) Curves of equal latitude are traces in the planes $z = k$. What is the shape of these curves?
(c) Meridians (curves of equal longitude) are traces in planes of the form $y = mx$. What is the shape of these meridians?

48. A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet (see the photo on page 856). The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.

49. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$ and $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)

50. Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.

51. Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leqslant 1.2$, $|y| \leqslant 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the xy-plane is an ellipse.

12 Review

Concept Check

1. What is the difference between a vector and a scalar?
2. How do you add two vectors geometrically? How do you add them algebraically?
3. If **a** is a vector and c is a scalar, how is $c\mathbf{a}$ related to **a** geometrically? How do you find $c\mathbf{a}$ algebraically?
4. How do you find the vector from one point to another?
5. How do you find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?
6. How are dot products useful?
7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate with diagrams.
8. How do you find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?
9. How are cross products useful?
10. (a) How do you find the area of the parallelogram determined by **a** and **b**?
    (b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?
11. How do you find a vector perpendicular to a plane?
12. How do you find the angle between two intersecting planes?
13. Write a vector equation, parametric equations, and symmetric equations for a line.
14. Write a vector equation and a scalar equation for a plane.
15. (a) How do you tell if two vectors are parallel?
    (b) How do you tell if two vectors are perpendicular?
    (c) How do you tell if two planes are parallel?
16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.
    (b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.
17. (a) How do you find the distance from a point to a line?
    (b) How do you find the distance from a point to a plane?
    (c) How do you find the distance between two lines?
18. What are the traces of a surface? How do you find them?
19. Write equations in standard form of the six types of quadric surfaces.

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = \langle u_1 v_1, u_2 v_2 \rangle$.
2. For any vectors **u** and **v** in V_3, $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.
3. For any vectors **u** and **v** in V_3, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.
4. For any vectors **u** and **v** in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.
5. For any vectors **u** and **v** in V_3, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
6. For any vectors **u** and **v** in V_3, $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.
7. For any vectors **u** and **v** in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.
8. For any vectors **u** and **v** in V_3 and any scalar k, $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$.
9. For any vectors **u** and **v** in V_3 and any scalar k, $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$.
10. For any vectors **u**, **v**, and **w** in V_3, $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.