METU

Northern Cyprus Campus

Math 120	Calculus for functions of several variables		Final Exam 03.06.2013
Last Name : Name : Student No:	Dept./Sec. : Time : 09 : 00 Duration : 120 *minutes*		Signature
6 QUESTIONS ON 6 PAGES			TOTAL 100 POINTS

1	2	3	4	5	6				

Solution

1. *(15=2+3+10 pts)* Consider the function $f(x,y) = 1 + x^2 + y^2$ in two independent variables x and y.

(a) Find the domain of f. f is defined for all $(x,y) \in \mathbb{R}^2$, so its domain is $\mathbb{R}^2$

(b) Find the largest set on which f is differentiable. $f_x = 2x$ $f_y = 2y$

Both f_x and f_y exists on $\mathbb{R}^2$ and ~~differentiable~~ continuous on $\mathbb{R}^2$, therefore f is differentiable on $\mathbb{R}^2$.

(c) Write the equation of the tangent plane to the graph of f at the point $(1,4)$.

$f_x(1,4) = 2x\big|_{(1,4)} = 2$ $\quad f_y(1,4) = 2y\big|_{(1,4)} = 8 \quad f(1,4) = 18$

Tangent plane at $(1,4,18)$ is given by the equation

$$f_x(1,4)(x-1) + f_y(1,4)(y-4) - (z - f(1,4)) = 0$$

that is

$$2(x-1) + 8(y-4) - (z-18) = 0.$$

2. *(15 pts)* The following parts use Green's theorem.

(a) Use **Green's theorem** to convert the following line integral to a double integral.

$$I = \oint_C (\cos(x) + 2x\arctan(y))dx + \left(xy + \frac{x^2}{1+y^2}\right) dy$$

where C is goes clockwise around the circle $x^2 + y^2 = 1$.

[Green's thm: ~~If~~ C ~~is~~ a POSITIVELY oriented, simple closed curve and D is bounded s.t. $\partial D = C$ and if P, Q have continuous partials then $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$]

In this integral C is negatively oriented.

$$Q_x = y + \frac{2x}{1+y^2} \qquad P_y = \frac{2x}{1+y^2}$$

By Green's theorem $I = -\iint_D (Q_x - P_y)\,dA = \iint_{\{x^2+y^2\leq 1\}} -y\,dA$.

(b) Use **Green's theorem** to convert the double integral $\iint_D y\,dA$ to a line integral of the form $\oint f(x,y)\,dy$, where D is the region $x^2 + y^2 \leq 1$.

Let $Q = xy$ and $P = 0$, then $Q_x = y$ and $P_y = 0$

P, Q ~~have~~ have continuous partials ~~near~~ and on D which is bounded.

By Green's thm, $\iint_D y\,dA = \iint_D Q_x - P_y\,dA = \oint_C xy\,dy$

where C is the unit circle with counterclockwise orientation.

3. *(15 pts)* Determine if the given series are convergent or not.

(a) $\sum_{n=1}^{\infty} \frac{n}{\ln(n)}$

$$\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \underset{\substack{\text{by l'Hopitals}\\ \text{rule}}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty$$

By divergence test $\sum \frac{n}{\ln n}$ is divergent.

(b) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n!}$

$$0 \leq \frac{\cos(n)}{n!} \leq \frac{1}{n!} \qquad \sum_{n=0}^{\infty} \frac{1}{n!} \text{ converges to } e$$

By comparison test $\sum_{n=1}^{\infty} \frac{\cos n}{n!}$ is ~~absolute~~ absolutely convergent

By absolute convergence test it is convergent.

(c) $\sum_{n=1}^{\infty} \frac{n+1}{n^4 - 13n^2 + 16n - 100}$

$\frac{n+1}{n^4-13n^2+16n-100}$ is positive at the tail part.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by p-test and $\lim_{n\to\infty} \frac{\frac{n+1}{n^4-13n^2+16n-100}}{\frac{1}{n^3}} = 1$

By limit comparison test, the series is convergent.

(d) $\sum_{n=1}^{\infty}((-1)^n + \frac{4}{n})$

$$\lim_{n\to\infty} (-1)^{2n} + \frac{4}{2n} = \lim_{n\to\infty} 1 + \frac{4}{2n} = 1$$

$$\lim_{n\to\infty} (-1)^{2n+1} + \frac{4}{2n+1} = \lim_{n\to\infty} (-1) + \frac{4}{2n+1} = -1 \neq 1$$

So $(-1)^n + \frac{4}{n}$ does not have a limit. By divergence test the series is divergent.

4. *(20 pts)* Find the interval I and the radius R of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n^{3/2}} (3x+5)^n .$$

Don't forget the boundary points of the interval I to be investigated.

$$\lim_{n\to\infty} \frac{\left|\frac{\ln^2(n+1)}{(n+1)^{3/2}}(3x+5)^{n+1}\right|}{\left|\frac{\ln^2 n}{n^{3/2}}(3x+5)^n\right|} = \lim_{n\to\infty} \underbrace{\left(\frac{\ln(n+1)}{\ln n}\right)^2 \left(\frac{n}{n+1}\right)^{3/2}}_{=1} |3x+5| = |3x+5|$$

By ratio test the series converges absolutely if $|3x+5|<1$

thatis if $-2<x<\frac{-4}{3}$, radius of convergence R is $\frac{1}{3}$

Check -2 and $\frac{-4}{3}$:

$x=\frac{-4}{3}$, we have $\sum_{n=1}^{\infty} \frac{\ln^2 n}{n^{3/2}}$ which is a positive series

$$\lim_{n\to\infty} \frac{\ln n}{n^{1/8}} = 0 \Rightarrow \ln n < n^{1/8} \text{ for large } n \Rightarrow \ln^2 n < n^{1/4}$$

$$\Rightarrow \frac{\ln^2 n}{n^{3/2}} < \frac{1}{n^{3/2-1/4}} = \frac{1}{n^{5/4}}$$

By p-test $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ is convergent.

By ~~Comparison~~ comparison test $\sum_{n=1}^{\infty} \frac{\ln^2 n}{n^{3/2}}$ is convergent.

$x=-2$, we have $\sum_{n=1}^{\infty} (-1)^n \frac{\ln^2 n}{n^{3/2}}$ which is absolutely convergent and therefore convergent.

So I, the interval of convergence is $[-2, \frac{-4}{3}]$

5. *(15 pts)* Find the power series expansion of the function $f(x) = \dfrac{x^2+1}{x}$ about the point $a = 1$. Don't use Taylor's formula for the coefficients.

$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ for $|t|<1$. $\quad \frac{x^2+1}{x} = x + \frac{1}{x} = 1 + (x-1) + \frac{1}{x}$

$$\frac{1}{x} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{for } 0<x<2$$

$$\frac{x^2+1}{x} = 1 + (x-1) + \sum_{n=0}^{\infty} (-1)^n (x-1)^n = 2 + \sum_{n=2}^{\infty} (-1)^n (x-1)^n \text{ for } x \in (0,2).$$

5⁺. Bonus. *(10 pts)* Let $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 3}$, $n \geq 1$, be a recursively defined sequence of numbers: $1, 2, \sqrt{5}, \sqrt{\sqrt{5}+3}, \ldots$. Show that $a_n \leq 3$ for all n, and investigate whether there is a limit L of the sequence or not? If the limit L does exist, find it; but if not, explain why.

(1) a_n is increasing by induction: $n=1$ $a_2 = 2$ $a_1 = 1 \Rightarrow a_{n+1} > a_n$.

~~Let~~ Assume $a_{n+1} > a_n$ for all $n < k$ ~~...~~

$$a_{k+1} = \sqrt{a_k+3} = \sqrt{\sqrt{a_{k-1}+3}+3} > \sqrt{a_{k-1}+3} = a_k$$

$\underbrace{>}_{\text{by induction hypothesis}}$ so a_n is increasing.

(2) $a_n \leq 3$ $\forall n$. ~~Assume $\exists n$~~ Assume $\exists$ such n and let n be the first element s.t. $a_{n+1} > 3 \Rightarrow \sqrt{a_n+3} > 3 \Rightarrow a_n + 3 > 9$

$\Rightarrow a_n > 6$ which is not possible by assumption.

(3) a_n is increasing and bounded above by $3 \Rightarrow a_n$ has a limit.

(4) Let $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} a_{n+1} = a$. $\quad a = \sqrt{a+3} \Rightarrow a = \frac{1-\sqrt{13}}{2} \approx -1.3$

this cannot be the limit.

or $a = \frac{1+\sqrt{13}}{2} \approx 2.3 \Rightarrow$ limit is $\frac{1+\sqrt{13}}{2}$. $\Leftarrow$

6. *(20=10+10 pts)* The following parts use Taylor's formula for power series representations.

(a) Use **Taylor's formula** to find the power series representation of

$$f(x) = x^4 - 2x^3 + x^2 - 1$$

around $a = 2$.

$f^{(0)}(2) = f(2) = \cancel{3}\ 3 \quad f'(2) = 12 \quad , f''(2) = 26$

$(f'(x) = 4x^3 - 6x^2 + 2x) \quad (f''(x) = 12x^2 - 12x + 2)$

$f'''(2) = 24x - 12\Big|_2 = 36 \qquad f^{(4)}(2) = 24 \qquad f^{(5)}(2) = 0$

$k>5 \quad f^{(k)}(2) = 0$

Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$

$$= \frac{3}{0!} + \frac{12}{1!}(x-2) + \frac{26}{2!}(x-2)^2 + \frac{36}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4 + 0$$

$$= 3 + 12(x-2) + 13(x-2)^2 + 6(x-2)^3 + (x-2)^4 \quad \text{on } \mathbb{R}.$$

(b) You should know the power series representation of

$$g(x) = \arctan(-2x)$$

around $a = 0$ without using Taylor's formula. Use this power series representation and Taylor's formula to find $g^{(2013)}(0)$ (the 2013th derivative of $\arctan(-2x)$ evaluated at 0).

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{on } (-1, 1).$$

$$\arctan(-2x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2^{2n+1}}{2n+1} x^{2n+1} \quad \text{on } \left(\frac{-1}{2}, \frac{1}{2}\right)$$

2013th term coefficient is $\dfrac{g^{(2013)}(0)}{2013!}$

$$\Rightarrow g^{(2013)}(0) = \frac{2013!\, 2^{6027}}{6027}.$$