METU - NCC

CALCULUS with ANALYTIC GEOMETRY FINAL EXAM		
Code : *MAT 119* Acad.Year: *2014-2015* Semester : *FALL* Date : *12.1.2015* Time : *9:00* Duration : *150 min*	Last Name: Name : Student # : Signature : KEY	List #:
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1. (15)	2. (25)	3. (24)	4. (20)	5. (16)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(pts)* Calculate the following limits. Please! explain your work.

(a) $\lim_{x\to 1^+} \frac{\ln(x^3)}{(\ln x)^3}$ $\left[\frac{0}{0}\right] \underset{\text{l'Hôpital's rule}}{=} \lim_{x\to 1^+} \frac{\frac{3x^2}{x^3}}{3(\ln x)^2 \cdot \frac{1}{x}} = \lim_{x\to 1^+} \frac{1}{(\ln x)^2} = \boxed{+\infty}$

(b) $\lim_{x\to 0^+} (1-\cos x)^{\tan(x)}$ $\left[\frac{0}{0}\right] = \lim_{x\to 0^+} e^{\ln(1-\cos x)^{\tan x}} = e^{\lim_{x\to 0^+} \tan x \ln(1-\cos x)} = e^{0} \boxed{=1}$

$\lim_{x\to 0^+} \tan x \cdot \ln(1-\cos x) = \lim_{x\to 0^+} \frac{\ln(1-\cos x)}{\cot x} \underset{\text{l'Hôpital's rule}}{=} \lim_{x\to 0^+} \frac{\frac{\sin x}{1-\cos x}}{\frac{-1}{\sin^2 x}}$

$[0.-\infty]$ $[\infty/\infty]$

$= \lim_{x\to 0^+} \frac{-\sin^3 x}{1-\cos x}$ $[0/0]$ $\underset{\text{l'Hôpitals rule}}{=} \lim_{x\to 0^+} \frac{-3\sin^2 x \cdot \cos x}{\sin x} = \lim_{x\to 0^+} -3\sin x \cdot \cos x = -3.0.1 \boxed{=0}$

(c) $\lim_{x\to 1} \frac{x^x - 1}{x-1}$ $\left[\frac{0}{0}\right] \underset{\text{l'Hôpital's rule}}{=} \lim_{x\to 1} \frac{\frac{d}{dx}(x^x-1)}{\frac{d}{dx}(x-1)}$ $\Rightarrow$ Need to compute $\frac{d}{dx}(x^x-1)$

Let $y = x^x - 1$

$y+1 = x^x \rightsquigarrow \ln(y+1) = \ln x^x = x.\ln x$ differentiate both sides

$\frac{y'}{y+1} = 1.\ln x + x.\frac{1}{x} = \ln x + 1 \rightsquigarrow y' = x^x(\ln x + 1)$

$\Rightarrow \lim_{x\to 1} \frac{x^x-1}{x-1} = \lim_{x\to 1} \frac{x^x(\ln x+1)}{1} = 1.(0+1) = \boxed{1}$

2. *(pts)* Compute the following integrals. Show your work.

(a) $\int e^{x+e^x}\,dx$ $= \int e^{e^x} \cdot e^x\,dx$ Put $u = e^x$, $du = e^x dx$

$= \int e^u\,du = e^u + C = \boxed{e^{e^x} + C}$

(b) $\int \sin^5 t \cos^4 t\,dt$ $= \int \sin^4 t \cos^4 t \sin t\,dt = \int (1-\cos^2 t)^2 \cos^4 t \sin t\,dt$

$\left.\begin{array}{l} u = \cos t \\ du = -\sin t\,dt \end{array}\right\}$ $= -\int (1-u^2)^2 u^4\,du = -\int (u^4 - 2u^6 + u^8)\,du$

$= -\frac{u^5}{5} + \frac{2}{7}u^7 - \frac{u^9}{9} + C = \boxed{-\frac{\cos^5 t}{5} + \frac{2}{7}\cos^7 t - \frac{\cos^9 t}{9} + C}$

(c) $\int \arctan(\sqrt{x})\,dx$ Substitution: $y = \sqrt{x}$, $dy = \frac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2y\,dy$

$= 2\int y \arctan(y)\,dy$ integ. by parts: $u = \arctan(y)$ $dV = y\,dy$

$du = \frac{1}{1+y^2}$ $V = \frac{y^2}{2}$

$= y^2 \arctan(y) - \int \frac{y^2}{1+y^2}\,dy = y^2 \arctan(y) - \int \left[1 - \frac{1}{1+y^2}\right] dy$

$= y^2 \arctan(y) - y + \arctan(y) + C = x\arctan(\sqrt{x}) - \sqrt{x} + \arctan(\sqrt{x}) + C$

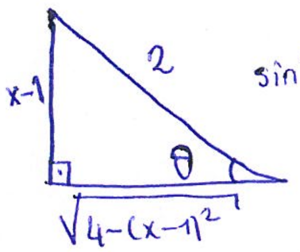
$\boxed{= (x+1)\arctan(\sqrt{x}) - \sqrt{x} + C}$

(d) $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x-1)^2}\,dx$ $\quad x-1 = 2\sin\theta \quad \left(\theta = \sin^{-1}\left(\frac{x-1}{2}\right)\right)$

$dx = 2\cos\theta\, d\theta$

$$= \int \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta = 4\int \cos\theta\cos\theta\, d\theta = 4\int \cos^2\theta\, d\theta = \frac{4}{2}\int (\cos 2\theta + 1)\, d\theta$$

$$= 2\left(\frac{1}{2}\sin 2\theta + \theta\right) + C = \boxed{\frac{(x-1)\sqrt{4-(x-1)^2}}{2} + \sin^{-1}\left(\frac{x-1}{2}\right) + C}$$



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{x-1}{2}\cdot\frac{\sqrt{4-(x-1)^2}}{2}$$

(e) $\int \frac{1}{(x-2)(x^2+4)}\,dx = I$

$$\frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \Rightarrow 1 = (A+B)x^2 + (-2B+C)x + 4A - 2C$$

$$\left.A+B=0,\ -2B+C=0,\ 4A-2C=1\ \right\}\ A = \frac{1}{8},\ B = \frac{-1}{8},\ C = \frac{-1}{4}$$

$$I = \frac{1}{8}\int \frac{dx}{x-2} - \int \frac{\frac{1}{8}x + \frac{1}{4}}{x^2+4}\,dx = \frac{1}{8}\ln|x-2| - \underbrace{\frac{1}{16}\int \frac{2x\,dx}{x^2+4}}_{\substack{u = x^2+4 \\ du = 2x\,dx}} - \frac{1}{4}\int \frac{dx}{x^2+4}$$

$$\boxed{I = \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

3. *(pts)* This problem has two unrelated parts.

(a) Evaluate the given integral or show that it is divergent.

(i) $\int_2^\infty \frac{1}{x\ln x}\,dx \;=\; \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\,dx$

Type 1 improper integral

$u=\ln x \qquad x=2 \quad u=\ln 2$

$du=\frac{dx}{x} \qquad x=t \quad u=\ln t$

$$=\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u} = \lim_{t\to\infty}\ln u\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty}\left[\ln(\ln t)-\ln(\ln 2)\right] = +\infty$$

$\Rightarrow$ $\boxed{\text{I divergent}}$

(ii) $\int_{-1}^{1}\frac{1}{x^2-2x}\,dx$

discont. at $x=0$, type 2

$$\frac{1}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}, \quad 1=(A+B)x-2A$$

$A=-1/2\,,\ B=1/2$

$$I=\frac{1}{2}\left[\int_{-1}^{0}\left(\frac{1}{x-2}-\frac{1}{x}\right)dx+\int_{0}^{+1}\left(\frac{1}{x-2}-\frac{1}{x}\right)dx\right]$$

$$=\frac{1}{2}\left[\lim_{t\to 0^-}\int_{-1}^{t}\left(\frac{1}{x-2}-\frac{1}{x}\right)dx+\lim_{t\to 0^+}\int_{t}^{1}\left(\frac{1}{x-2}-\frac{1}{x}\right)dx\right]=\frac{1}{2}\left[\lim_{t\to 0^-}\left(\ln|x-2|-\ln|x|\right)\Big|_{-1}^{t}+\lim_{t\to 0^+}\left(\ln\left|\frac{x-2}{x}\right|\right)\Big|_{t}^{1}\right]$$

Since $\lim_{t\to 0^-}\left(\ln|t-2|-\ln|t|-\ln 3+\ln 1\right)=+\infty$, $\boxed{\text{I is divergent}}$

(b) Use the Comparison Theorem to determine whether the following integral is convergent or divergent.

$\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

$$\frac{2+e^{-x}}{x}=\frac{2}{x}+\underbrace{\frac{e^{-x}}{x}}_{>0 \text{ for all } x>1} > \frac{2}{x}$$

$$\int_1^\infty\frac{2}{x}dx=2\int_1^\infty\frac{1}{x}dx=2\lim_{t\to\infty}\ln x\Big|_1^t=2\lim_{t\to\infty}\left(\ln t-0\right)=+\infty$$

$\Rightarrow$ Since $\frac{2+e^{-x}}{x}>\frac{2}{x}$ and $\int_1^\infty\frac{2}{x}dx$ is divergent, $\boxed{\int_1^\infty\frac{2+e^{-x}}{x}dx \text{ is also divergent.}}$

Name: ID:

4. *(pts)* Suppose that $x \geqslant 0$, $y \geqslant 0$ and $x+y=6$. Find the values of x and y which maximize $4x+y^2$.

$x = 6-y \rightsquigarrow f(y) = 4.(6-y)+y^2 = 24-4y+y^2, \quad 0 \leqslant y \leqslant 6$

f might take its max. values at the critical numbers and at the end points:

$f'(y) = -4+2y \rightsquigarrow f'(2)=0 \rightsquigarrow f(2) = 24-4.2+2^2 = 20$

$f(0) = 24, \quad f(6) = 24-24+36 = 36$

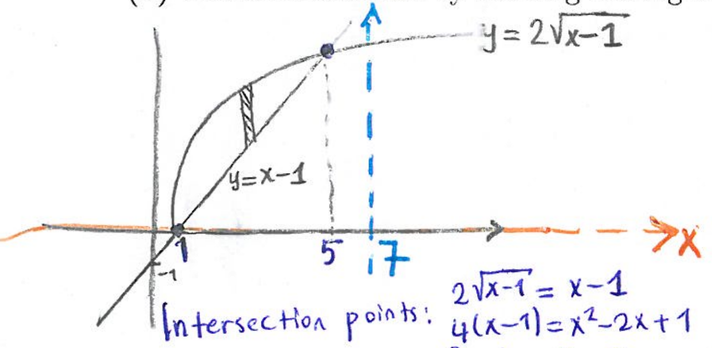
y	0		2		6
f'		$-$	0	$+$	
f	24	$\searrow$	20 (minimum value)	$\nearrow$	36

$\Rightarrow f(6) = 36$ is the maximum val.

$\boxed{y=6, \ x=0}$

5. Let R be the region bounded by the curves $y = 2\sqrt{x-1}$ and $y = x-1$. Write the integrals which calculate the volumes of the solids described below. Do not evaluate the integrals.

(a) The solid obtained by rotating the region R around the line $x = 7$.



Intersection points: $2\sqrt{x-1} = x-1$
$4(x-1) = x^2-2x+1$
$x^2-6x+5=0$

$\Rightarrow x=1, \ x=5$

Cylindrical Shell Method:

$$V = 2\pi \int_1^5 (7-x)\left(2\sqrt{x-1}-(x-1)\right) dx$$

(b) The solid obtained by rotating the region R around the x-axis.

Disk / Washer Method:

$$V = \pi \int_1^5 \left[4(x-1)-(x-1)^2\right] dx$$

6. *(pts)* Given $f(x) = xe^{-2x^2}$.

(a) Find the domain, x-intercepts and y-intercept of $f(x)$. $\boxed{\text{domain} = \mathbb{R} = (-\infty, \infty)}$

$xe^{-2x^2} = 0 \rightarrow x = 0$: x-intercepts & the y intercept is the $\boxed{\text{origin } (0,0)}$

(b) Find the asymptotes of $f(x)$.

no Vertical Asymp. since domain $= \mathbb{R}$

Horizontal Asymp. : $\lim\limits_{x\to-\infty} \dfrac{x}{e^{2x^2}} \underset{\text{l'Hôpital}}{=\!=} \lim\limits_{x\to-\infty} \dfrac{1}{4xe^{2x^2}} = 0$ $\boxed{y = 0 \text{ H.A.}}$

$\lim\limits_{x\to+\infty} \dfrac{x}{e^{2x^2}} \underset{\text{l'Hôpital}}{=\!=} \lim\limits_{x\to+\infty} \dfrac{1}{4xe^{2x^2}} = 0$

(c) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$f'(x) = e^{-2x^2} + x.(-4x).e^{-2x^2} = e^{-2x^2}(1-4x^2) = e^{-2x^2}(1-2x)(1+2x)$

x	$-\infty$	$-1/2$	$1/2$	$+\infty$
f'	$-$	ϕ $+$	ϕ $-$	
f	decrease	increase	decrease	

$\boxed{f(-\frac{1}{2}) = \frac{-1}{2\sqrt{e}}}$ loc. minimum ; $\boxed{f(\frac{1}{2}) = \frac{1}{2\sqrt{e}}}$ local maximum

(d) Find the intervals of concavity and inflection points of $f(x)$.

$f''(x) = -4xe^{-2x^2}(1-4x^2) + (-8x).e^{-2x^2} = 4x(4x^2-3)e^{-2x^2} = 4x(2x-\sqrt{3})(2x+\sqrt{3})e^{-2x^2}$

x	$-\infty$	$-\sqrt{3}/2$	0	$\sqrt{3}/2$	∞
f''	$-$	ϕ $+$	ϕ $-$	ϕ $+$	
f	concave down	conc. up	conc. down	conc. up	

$\boxed{f(-\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{2e^{3/2}}}$, $\boxed{f(0) = 0}$, $\boxed{f(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2e^{3/2}}}$ are the inflection points.

(e) Sketch the graph of $f(x)$. **Don't forget** to indicate the intercepts, local maximum/minimum and inflection points on your graph, if there are any.