4. *(5×4=20pts)* Evaluate the following integrals.

(A) $\int_1^2 \frac{x^2+1}{\sqrt{x}}\,dx = \int_1^2 (x^{3/2}+x^{-1/2})\,dx = \frac{2}{5}x^{5/2}+2x^{1/2}\Big|_1^2$

$= \left(\frac{2}{5}\cdot 2^{5/2}+2\cdot 2^{1/2}\right)-\left(\frac{2}{5}+2\right)$

(B) $\int_0^3 \sqrt{21-7x}\,dx = -\frac{1}{7}\int_{21}^{0}\sqrt{u}\,du = -\frac{1}{7}\cdot\frac{2}{3}u^{3/2}\Big|_{21}^{0} = +\frac{2}{21}\,21^{3/2}$

$u=21-7x \Rightarrow du=-7\,dx$

$u(0)=21$

$u(3)=0$

(C) $\int_{-1}^1 (x^2\sin(x)+3)\,dx = \int_{-1}^1 x^2\sin x\,dx+\int_{-1}^1 3\,dx = 0+2.3=6$

$\left.\begin{array}{l} f(x)=x^2\sin x \\ f(-x)=(-x)^2\sin(-x)=-x^2\sin x \end{array}\right\} \Rightarrow$ So $f(x)$ is odd

(D) $\int x^5(x^3+1)^{1/3}\,dx = \int x^3(x^3+1)^{1/3}\cdot x^2\,dx = \frac{1}{3}\int (u-1)u^{1/3}\,du = \frac{1}{3}\int (u^{4/3}-u^{1/3})\,du$

$u=x^3+1 \Rightarrow du=3x^2dx$

$\frac{1}{3}du = x^2.dx$

$\downarrow$

$x^3=u-1$

$=\frac{1}{3}\left(\frac{3}{7}u^{7/3}-\frac{3}{4}u^{4/3}\right)+C$

$=\frac{1}{3}\left(\frac{3}{7}(x^3+1)^{7/3}-\frac{3}{4}(x^3+1)^{4/3}\right)+C$

METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 2	
Code : *MAT 119*	Last Name:
Acad.Year: *2013-2014*	Name :
Semester : *FALL*	Student # :
Date : *07.12.2013*	Signature :
Time : *9:40*	6 QUESTIONS ON 6 PAGES
Duration : *110 min*	TOTAL 100 POINTS

1. (15)	2. (25)	3. (10)	4. (20)	5. (18)	6. (12)	

Please draw a box around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(15pts)* Compute the area between $x+y^2=0$ and $x+2y^2=4$ for $0\leqslant y\leqslant 4$.

$x=-y^2$ & $x=4-2y^2$

$A=\int_0^4 |-y^2-4+2y^2|\,dy = \int_0^4 |y^2-4|\,dy$

$y^2-4=0 \Rightarrow y=\mp 2$

		-2		2	
y^2-4	+	0	−	0	+

$\Rightarrow A=\int_0^4 |y^2-4|\,dy = \int_0^2(-y^2+4)\,dy+\int_2^4 (y^2-4)\,dy$

$A= -\frac{1}{3}y^3+4y\Big|_0^2 + \frac{1}{3}y^3-4y\Big|_2^4$

$A=\left(-\frac{8}{3}+8\right)+\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)$

2. *(1+2+4+6+6+6=25pts)* Let $f(x) = \dfrac{|x|-1}{x-2}$. In this question we will sketch the graph of $f(x)$.

(A) What is the domain of $f(x)$?

$\mathrm{Dom}(f) = \mathbb{R} - \{2\}$

(B) Find the intercepts of $f(x)$.

$f(0) = \frac{1}{2} \Rightarrow (0, \frac{1}{2})$ is the y-intercept

$0 = f(x) \Rightarrow |x|-1 = 0 \Rightarrow |x| = 1 \Rightarrow x = \mp 1$

So $(-1,0)$ & $(1,0)$ are the x-intercepts

(C) Find the asympote(s) of $f(x)$.

Vertical:

$\lim\limits_{x\to 2^+} \frac{|x|-1}{x-2} = \lim\limits_{x\to 2^+} \frac{x-1}{x-2} = \infty$ & $\lim\limits_{x\to 2^-} \frac{|x|-1}{x-2} = \lim\limits_{x\to 2^-} \frac{x-1}{x-2} = -\infty$

So $x=2$ is a vertical asymptote

Horizontal:

$\lim\limits_{x\to\infty} \frac{|x|-1}{x-2} = \lim\limits_{x\to\infty} \frac{x-1}{x-2} = 1$

$\lim\limits_{x\to -\infty} \frac{|x|-1}{x-2} = \lim\limits_{x\to -\infty} \frac{-x-1}{x-2} = -1$

So $y=1$ & $y=-1$ are horizontal asymptotes

(D) Find the intervals of increase/decrease. Indicate local max/min points.

$x>0$: $f(x) = \frac{x-1}{x-2} = 1 + \frac{1}{x-2} \Rightarrow f'(x) = -\frac{1}{(x-2)^2}$, $\boxed{x \neq 2}$

$x<0$: $f(x) = \frac{-x-1}{x-2} = -1 - \frac{3}{x-2} \Rightarrow f'(x) = \frac{3}{(x-2)^2}$

We know $f'(x)$ is undefined when $x=0$ since to the right & left of zero the derivatives do not agree when $x \to 0$.

	0		2	
$f'(x)$	+	−		−
$f(x)$	Inc	Dec		Dec

(E) Find the intervals of concavity. Indicate inflection points.

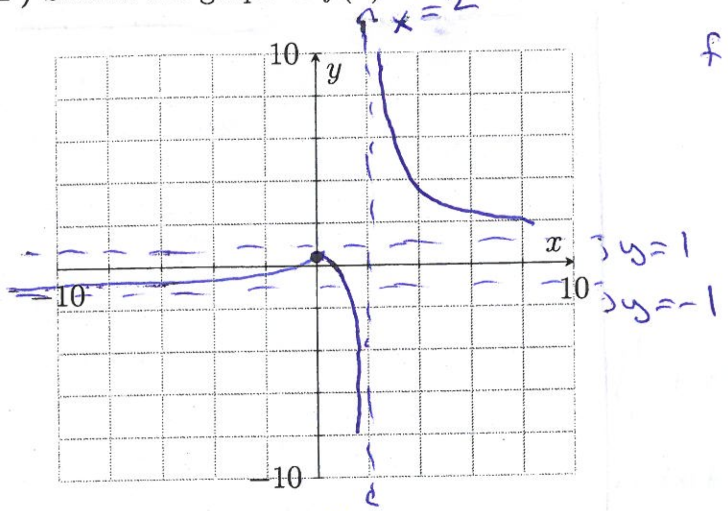
$f''(x) = \frac{2}{(x-2)^3}$ when $x>0$

$f''(x) = \frac{-6}{(x-2)^3}$ when <0

$f''(x)$ is undefined when $x=0$ & $x=2$

	0		2	
$f''(x)$	+	−		+
$f(x)$	C.U.	C.D.		C.U.

(F) Sketch the graph of $f(x)$.



$f(0) = 1/2$

3. *(10pts)* Suppose that $f(x)$ is a continuous and differentiable function with $f(3) = 5$ and $6 \leqslant f'(x) \leqslant 8$ for all x. Prove that $17 \leqslant f(5) \leqslant 21$.

By MVT

$6 \leq \frac{f(5)-f(3)}{5-3} \leq 8$

$12 \leq f(5) - f(3) \leq 16$

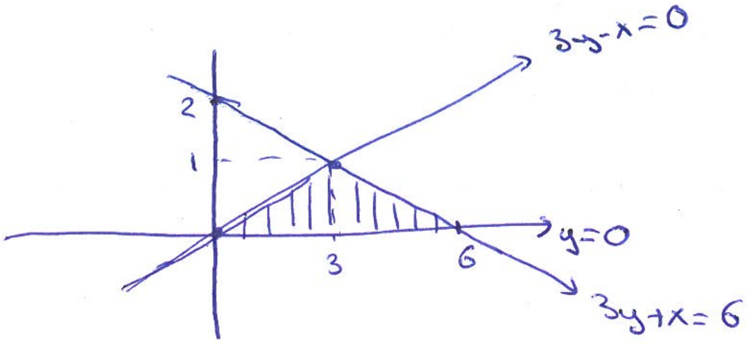
$12 \leq f(5) - 5 \leq 16$

$17 \leq f(5) \leq 21$

5. *(18pts)* Calculate the volume of a tent with triangular base bounded by the equations $y = 0$, $3y - x = 0$, and $3y + x = 6$; whose cross sections perpendicular to the base, parallel to the y-axis are semi-circles.

$3y - x = 0 \Rightarrow y = \frac{x}{3}$

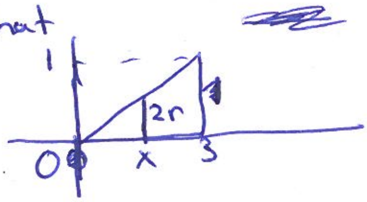
$3y + x = 6 \Rightarrow y = 2 - \frac{x}{3}$



Shaded region is the triangular base

Note that the volume for $x \in [0,3]$ & $x \in [3,6]$ of this solid are the same. So just find the volume on $x \in [0,3]$ & multiply by 2. For $x \in [0,3]$, the cross-section at x is:

Observe that



So $\frac{2r}{x} = \frac{1}{3}$

$\Rightarrow r = \frac{x}{6}$

So the area of the cross-section at x is

$$A(x) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \frac{x^2}{36} = \frac{1}{72}\pi x^2$$

Now

$$V = 2\int_0^3 \frac{1}{72}\pi x^2 \, dx = \frac{2\pi}{72}\left(\frac{1}{3}x^3\Big|_0^3\right) = \frac{\pi}{36}\cdot 9 = \frac{\pi}{4}$$

6. *(12pts)* Suppose that $f(x)$ is a differentiable function defined on $(0, \infty)$ and $x\, f'(x) > 2\, f(x)$ for all $x \in (0, \infty)$. Prove that the function $g(x) = \frac{f(x)}{x^2}$ has no global maximum on $(0, \infty)$.

$$g(x) = \frac{f(x)}{x^2} \Rightarrow g'(x) = \frac{f'(x)x^2 - f(x).2x}{x^4}$$

Since $x f'(x) > 2f(x)$ & $x \in (0,\infty)$ we have

$x^2 f'(x) > 2x f(x)$

& so $f'(x)x^2 - f(x)2x > 0$

Therefore $g'(x) > 0$ for all $x \in (0,\infty)$, which means $g(x)$ is increasing on $(0,\infty)$. So $g(x)$ has no global max on $(0,\infty)$