METU-NCC
Mathematics Group

	Calculus with Analytic Geometry First Midterm Exam	
Code : MATH 119 Acad.Year : 2011–2012 Semester : Spring Coord. : S.D./H.T. Date : 24.03.2012 Time : 14.40 Duration : 120 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 7 Questions on 6 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5	Q6	Q7		

<u>Q.1</u> (5 + 10 = 15 pts) Let $f(x) = |x-1|x^2 + \sin(x-1)$.

(a) Is f continuous at $x = 1$?

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)x^2 + \sin(x-1) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x)x^2 + \sin(x-1) = 0$$

So $\lim_{x\to 1} f(x) = 0 = f(1)$ & $f(x)$ is cont. at $x=1$

(a) Is f differentiable at $x = 1$? If yes, find $f'(1)$.

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{|h|(1+h^2) + \sin(h)}{h}$$

$$= \lim_{h\to 0^+} \left[\frac{h(1+h^2)}{h} + \frac{\sin(h)}{h}\right] = 2$$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{|h|(1+h^2) + \sin(h)}{h}$$

$$= \lim_{h\to 0^-} \left[\frac{-h(1+h^2)}{h} + \frac{\sin(h)}{h}\right] = 0$$

So $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ does not exist.

Q.2 ($4 \times 5 = 20$ pts) Evaluate the following limits:

(a) $\lim\limits_{\theta \to 0} \dfrac{5\theta}{\sin\theta + \theta\tan\theta}$

$= \lim\limits_{\theta \to 0} \dfrac{5}{\frac{\sin\theta}{\theta} + \frac{\theta\tan\theta}{\theta}} = 5$

(b) $\lim\limits_{x \to 0} \dfrac{2\cos x}{\cos(2x) + \tan x} = \dfrac{2 \cdot \cos(0)}{\cos(0)} = 2$

(c) $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 3}{\sqrt{x^2 - 2x + 1}} = \lim\limits_{x \to 1} \dfrac{(x+3)(x-1)}{\sqrt{(x-1)^2}} = \lim\limits_{x \to 1} \dfrac{(x+3)(x-1)}{|x-1|}$

Consider

$\lim\limits_{x \to 1^+} \dfrac{(x+3)(x-1)}{|x-1|} = \lim\limits_{x \to 1^+} \dfrac{(x+3)(x-1)}{(x-1)} = 4$

$\lim\limits_{x \to 1^-} \dfrac{(x+3)(x-1)}{|x-1|} = \lim\limits_{x \to 1^-} \dfrac{(x+3)(x-1)}{-(x-1)} = -4$

So the above limit does not exist.

(d) $\lim\limits_{x \to 2} \dfrac{x^3 + 2x^2 - 8x}{x^2 - 7x + 10} = \lim\limits_{x \to 2} \dfrac{x(x^2+2x-8)}{(x-5)(x-2)}$

$= \lim\limits_{x \to 2} \dfrac{x(x+4)(x-2)}{(x-5)(x-2)} = \dfrac{12}{-3} = -4$

(e) $\lim\limits_{x \to -1} (x+1)^3 \cos\left(\dfrac{1}{x+1}\right)$

$-1 \le \cos\left(\frac{1}{x+1}\right) \le 1 \Rightarrow -(x+1)^3 \le \cos\left(\frac{1}{x+1}\right)(x+1)^3 \le (x+1)^3$

Since $\lim\limits_{x \to -1} -(x+1)^3 = 0 = \lim\limits_{x \to -1} (x+1)^3$, by the squeeze theorem we obtain $\lim\limits_{x \to -1} (x+1)^3 \cos\left(\frac{1}{x+1}\right) = 0$

Q.3 ($5 \times 4 = 20$ pts)

(a) If $f(x) = \dfrac{\sqrt{x}\sin x}{\cos(x^2)}$, find $f'(x)$.

$$f'(x) = \frac{\left(\frac{1}{2}x^{-1/2} \cdot \sin x + \sqrt{x}\cos x\right) \cdot \cos(x^2) - \sqrt{x}\sin x \cdot (-\sin(x^2))\, 2x}{\cos^2(x^2)}$$

(b) If $h(x) = \dfrac{f(x)}{g(x)}$ and $f(0) = 5,\ f'(0) = 6,\ g(0) = 8$ and $g'(0) = 4$, find $h'(0)$.

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \Rightarrow h'(0) = \frac{6.8 - 5.4}{64}$$

$$h'(0) = \frac{28}{64}$$

(c) If $y^3 - y^2 = x + y\sin x$, find $\dfrac{dy}{dx}$.

$$3y^2 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}\sin x + y\cos x$$

$$\frac{dy}{dx}\left(3y^2 - 2y - \sin x\right) = 1 + y\cos x$$

$$\frac{dy}{dx} = \frac{1 + y\cos x}{3y^2 - 2y - \sin x}$$

(d) Find equations of the tangent and normal to the curve in Part (c) at the point (0,1).

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{2}{1} = 2 \Rightarrow \text{Tangent line } y - 1 = 2(x-0)$$

$$\text{Normal line } y - 1 = -\frac{1}{2}(x-0)$$

Q.4 (3 + 7 = 10 pts) Consider the function $f(x) = 2x - 3x^{2/3}$ on the interval $x \in [-1, 1]$.

(a) Explain why this function has indeed an absolute maximum and a minimum value.

f(x) is a cont. function & so ~~it~~ has a max/min on the closed interval [-1,1]

(b) Find maximum and minimum values.

$$f'(x) = 2 - \frac{2}{x^{1/3}} = \frac{2x^{1/3} - 2}{x^{1/3}}$$

Crit Points: $x = 0$ & $x = 1$

$f(-1) = -2 - 3 = -5 \leftarrow$ MIN

$f(0) = 0 \leftarrow$ MAX

$f(1) = 2 - 3 = -1$

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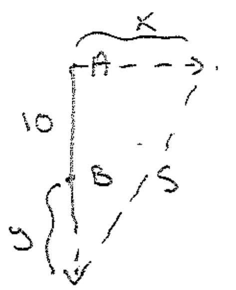
Q.5 (15 pts) At some time, Athlete A is $10m$ due NORTH of Athlete B. If A is running east at a constant speed of $8m/s$, and B is running south at $4m/s$, find the distance S between two athletes 5 seconds later. At this time, what is the rate of change of distance S? (In other words, how fast is the distance S between them increasing 5 seconds later?)

$$\frac{dx}{dt} = 8 \quad \frac{dy}{dt} = 4, \quad S(5) = ? \;\&\; \left.\frac{dS}{dt}\right|_{t=5} = ?$$



$$x(5) = 40 \qquad S^2 = x^2 + (y+10)^2$$

$$y(5) = 20$$

$$S(5)^2 = (40)^2 + (30)^2 \Rightarrow S(5) = 50$$

$$2S\frac{dS}{dt} = 2x \cdot \frac{dx}{dt} + 2(y+10) \cdot \frac{dy}{dt}$$

$$\Rightarrow 100 \left.\frac{dS}{dt}\right|_{t=5} = 80 \cdot 8 + 60 \cdot 4$$

$$\Rightarrow \left.\frac{dS}{dt}\right|_{t=5} = 8.8$$

Q.6 (10 pts) Show that there exists a real number, say x, whose fourth power is 3 more than itself.

We want to show $x^4 = x+3$, for some x.

Let $f(x) = x^4 - x - 3$. It suffices to find c such that $f(c)=0$.

$f(0) = -3 < 0$
$f(2) = 11 > 0$

Since $f(x)$ is cont, by the intermediate value therom, there is $c \in (0,2)$ such that $f(c)=0$.

Q.7 (10 pts) Prove that

$$\lim_{x\to 2} \frac{3x}{x-1} = 6$$

using ϵ, δ definition of limit.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow \left|\frac{3x}{x-1} - 6\right| < \varepsilon$$

Consider $\left|\frac{3x}{x-1} - 6\right| < \varepsilon$.

$$\Leftrightarrow \left|\frac{-3x+6}{x-1}\right| < \varepsilon \Leftrightarrow 3|x-2| \cdot \frac{1}{|x-1|} < \varepsilon.$$

Since $x \to 2$, we may assume $x \in (1.5, 2.5)$.
Then $0.5 < |x-1| < 1.5 \Rightarrow \frac{1}{|x-1|} < 2$.

Now choose $\delta = \frac{\varepsilon}{6}$. Then

$$|x-2| < \frac{\varepsilon}{6} \Rightarrow 3|x-2| < \frac{\varepsilon}{2} \Rightarrow 3|x-2| \cdot \frac{1}{|x-1|} < 6$$

Hence $\left|\frac{3x}{x-1} - 6\right| < \varepsilon$, as required.