METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL EXAM	
Code : *MAT 119* Acad.Year: *2012-2013* Semester : *FALL* Date : *14.1.2013* Time : *9:00* Duration : *180 minutes*	Last Name: Name : Department: Signature: Student No.: Section: 8 QUESTIONS ON 8 PAGES TOTAL 100 POINTS

1. (12)	2. (12)	3. (12)	4. (12)	5. (13)	6. (13)	7. (13)	8. (13)	Bonus.		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(pts)* Compute the following limits, if they exist.

(A) $\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \left(\sqrt{x^2+x}-x\right)\frac{\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)} = \lim_{x\to\infty} \frac{\cancel{(x^2}+x)-\cancel{(x^2)}}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x}}+1\right)} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$

(B) $\lim_{x\to 0^+} \frac{\sin^2 x}{\tan x - x}$ $\left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{2\sin x\cdot\cos x}{\sec^2 x - 1}$ $\left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{2\cos^2 x - 2\sin^2 x \to 2}{2\sec x\cdot\sec x\cdot\tan x \to 0} = +\infty.$

(C) $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin x} = \lim_{x\to 0^+} e^{\sin x\cdot\ln\left(\frac{1}{x}\right)} = e^{\lim_{x\to 0^+} -\sin x\cdot\ln x} = e^0 = 1$

$\lim_{x\to 0^+} -\sin x\cdot\ln x$ $(0\cdot(-\infty)) = \lim_{x\to 0^+} -\frac{\ln x}{\frac{1}{\sin x}}$ $\left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\csc x\cdot\cot x}$

$= \lim_{x\to 0^+} \frac{\sin x}{x}\cdot\frac{\sin x}{\cos x} = 1\cdot 0 = 0$

(D) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{xe^x}\right)$ $(\infty - \infty)$

$= \lim_{x\to 0^+} \frac{e^x - 1}{x\cdot e^x}$ $\left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x}{x\cdot e^x + e^x} = \frac{1}{0+1} = 1$

2. *(pts)* Compute the following derivatives.

(A) $\frac{d}{dx}\left(\arcsin(\sqrt{1-x^2})\right) = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$

$$= \frac{1}{\sqrt{x^2}} \cdot \frac{x}{\sqrt{1-x^2}} = \frac{x}{|x|} \cdot \frac{1}{\sqrt{1-x^2}}$$

(B) $\frac{d}{dx}\left(\sqrt{\frac{\ln x}{x}}\right) = \frac{\left(\frac{\ln x}{x}\right)'}{2\sqrt{\frac{\ln x}{x}}} = \frac{\frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2}}{2\sqrt{\frac{\ln x}{x}}} = \frac{1-\ln x}{2x^2\sqrt{\frac{\ln x}{x}}}$

(C) $\frac{d}{dx}\left(\underbrace{\frac{\sqrt[3]{x-2}\,\ln(2x+1)}{\tan^2(3x)(x^2-3x+1)^4}}_{u}\right) \Rightarrow \ln u = \frac{1}{3}\ln(x-2) + \ln(\ln(2x+1)) - 2\ln(\tan 3x) -$

$4\ln(x^2-3x$

$$\Rightarrow \frac{u'}{u} = \frac{1}{3}\,\frac{1}{x-2} + \frac{1}{\ln(2x+1)}\cdot\frac{1}{2x+1}\cdot 2 - 2\cdot\frac{1}{\tan 3x}\cdot\frac{1}{1+}$$

$$-4\cdot\frac{1}{x^2-3x+1}\,(2$$

$$\Rightarrow u' = \left(\frac{\sqrt[3]{x-2}\,\ln(2x+1)}{\tan^2(3x)\,(x^2-3x+1)^4}\right)\cdot\left[\frac{1}{3(x-2)} + \frac{2}{(2x+1)\ln(2x+1)} - \frac{6}{(1+9x^2)\tan(3x)} - \frac{4.(2x-3)}{x^2-3x+1}\right.$$

(D) Find $\frac{dy}{dx}$ at $(2,3)$ on the curve $y^x = y + 3x$.

$$\frac{d}{dx}(y^x) = \frac{d}{dx}(y+3x)$$

$$(y^x)' = \left(e^{x\cdot\ln(y)}\right)' = e^{x\cdot\ln(y)}\cdot\left(\ln(y) + \frac{x\cdot y'}{y}\right)$$

$$= y^x\cdot\left(\ln(y) + x\cdot\frac{y'}{y}\right)$$

$$y^x\left(\ln y + \frac{xy'}{y}\right) = y' + 3$$

$$y'\left(\frac{y^x\cdot x}{y} - 1\right) = 3 - y^x\cdot\ln y$$

$$\frac{dy}{dx} = y' = \frac{3 - y^x\cdot\ln y}{\frac{y^x\cdot x}{y} - 1} \Rightarrow \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3-3^2\ln(3)}{3^2\cdot\frac{2}{3} - 1}$$

3. *(pts)* Compute the following integrals.

(A) $\int 2^{lnx}\,dx \;=\!=\; \int 2^{u} \cdot e^{u} du = \int (2e)^{u} du = \frac{(2e)^{u}}{\ln(2 \cdot e)} + C$

$u = \ln x \Rightarrow x = e^{u}$

$du = \frac{1}{x} dx$

$x \cdot du = dx$

$e^{u} du = dx$

$= \frac{2^{u} \cdot e^{u}}{\ln 2 + \ln e} + C$

$= \frac{2^{\ln x} \cdot x}{\ln 2 + 1} + C$

(B) $\int \frac{\sqrt[3]{\arctan(x)+1}}{x^2+1}\,dx \;=\; \int \sqrt[3]{u}\,du = \frac{3u^{4/3}}{4} + C$

$u = \arctan x + 1$

$du = \frac{1}{1+x^2} dx$

$= \frac{3}{4} \sqrt[3]{(\arctan x + 1)^4} + C.$

(C) $\int \sin^2(2x)\cos(x)\,dx \;=\; \int (2\sin x \cdot \cos x)^2 \cos(x) dx = \int 4 \sin^2 x \cos^3 x\, dx$

$= \int 4 \sin^2 x \cdot (1 - \sin^2 x) \cdot \cos x dx = \int 4u^2(1-u^2) du$

$u = \sin x$

$= \int 4u^2 - 4u^4 du \qquad du = \cos x dx$

$= \frac{4u^3}{3} - \frac{4u^5}{5} + C = \frac{4}{3}\sin^3 x - \frac{4}{5}\sin^5 x + C$

(D) $\int \underbrace{(4x^3+2x)}_{dv} \underbrace{\arctan(x)}_{u} dx \;=\; (x^4+x^2) \cdot \arctan x - \int (x^4+x^2) \cdot \frac{1}{1+x^2} dx$

$4x^3 + 2x = dv \Rightarrow v = x^4 + x^2$

$\arctan x = u \Rightarrow du = \frac{1}{1+x^2} dx$

$= (x^4+x^2)\arctan x - \int x^2 dx$

$= (x^4+x^2)\arctan x - \frac{x^3}{3} + C$

4. *(pts)* Compute the following integrals.

(E) $\int \frac{1}{x\sqrt{x-4}}\,dx \quad = \int \frac{2u\,du}{(u^2+4)\cdot u} = 2\int \frac{du}{u^2+4} = \frac{2}{4}\int \frac{du}{(\frac{u}{2})^2+1}$

$u = \sqrt{x-4} \qquad du = \frac{1}{2\sqrt{x-4}}\,dx$

$u^2 = x-4$

$x = u^2+4 \qquad dx = 2u\,du$

$s = \frac{u}{2} \qquad ds = \frac{du}{2}$

$= \frac{2}{4}\int \frac{2\,ds}{s^2+1}$

$= \tan^{-1}(s) + C$

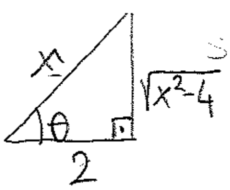
$= \tan^{-1}\left(\frac{\sqrt{x-4}}{2}\right) + C$

(F) $\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{4\sec^2\theta-4}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\cdot\sqrt{\sec^2\theta-1}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta}$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$



$\sec\theta = \frac{x}{2}$

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$

(G) $\int \frac{5x^2+7x}{(x-1)(x+1)^2}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{2}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx = 3\ln|x-1| + 2\ln|x+1| - \frac{1}{(x+1)} +$

$\frac{5x^2+7x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{Ax^2+2Ax+A+Bx^2-B+Cx-C}{(x-1)(x+1)^2}$

$(x+1)^2 \qquad (x^2-1) \qquad (x-1)$

$A+B=5$

$2A+C=7$

$A-B-C=0$

$(+) \Rightarrow$

$A+B=5$

$+\ 3A-B=7$

$4A=12$

$A=3$

$B=2$

5. *(pts)* Determine whether the following integrals converge or diverge. If they converge, find what they converge to.

(A) $\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{t\to\infty}\left[\int_0^t \frac{e^x}{e^{2x}+1}\,dx\right] = \lim_{t\to\infty}\left[\int_1^{e^t}\frac{du}{u^2+1}\right] = \lim_{t\to\infty}\left[\tan^{-1}(u)\Big|_1^{e^t}\right]$

$u = e^x$

$du = e^x dx$

$= \lim_{t\to\infty} \tan^{-1}(e^t) - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ Convergent !

(B) $\int_{-1}^1 \frac{1}{x^2+2x}\,dx = \int_{-1}^1 \frac{1}{x(x+2)}\,dx = \int_{-1}^0 \frac{1}{x(x+2)}\,dx + \int_0^1 \frac{1}{x(x+2)}\,dx$

$\int_0^1 \frac{1}{x(x+2)}\,dx = \boxed{+\frac{1}{2}\int_0^1 \frac{1}{x}\,dx} - \frac{1}{2}\int_0^1 \frac{1}{x+2}\,dx$

Divergent by p-test

Hence $\int_{-1}^1 \frac{1}{x(x+2)}\,dx$ is divergent !

(C) $\int_2^\infty \frac{3-\arctan x}{x^{2/3}-1}\,dx$ (*Hint: Use comparison theorem.*)

On $(1,\infty)$ $\quad 0 < \arctan x < \pi/2 < 2 \qquad x^{2/3} - 1 < x^{2/3}$

So, $\frac{3-\arctan x}{x^{2/3}-1} > \frac{1}{x^{2/3}}$

$\underbrace{\int_2^\infty \frac{3-\arctan x}{x^{2/3}-1}\,dx}_{\text{divergent by}} > \underbrace{\int_2^\infty \frac{1}{x^{2/3}}\,dx}_{\text{divergent by p-test}}$

6. *(12pts)* Sketch the graph of the function $f(x) = x^2e^x$ by follwing the guidelines below.

(A) Write down the domain of f, and find the intercepts with the axes.

$\mathrm{Dom}(f) = \mathbb{R}$

$f(0) = 0$ y-intercept

$0 = f(x) = x^2e^x \Rightarrow x = 0$ x-intercept

(B) Find the asymptotes if any.

No vertical asymptote since $\mathrm{Dom}(f) = \mathbb{R}$

$$\lim_{x\to\infty} x^2 \cdot e^x = \infty$$

$$\lim_{x\to-\infty} x^2 \cdot e^x \ (\infty \cdot 0) = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

$y = 0$ is a Horizontal Asymptote

(C) Find intervals of increase and decrease.

$$f'(x) = 2xe^x + x^2 \cdot e^x = xe^x(2+x) = 0 \Rightarrow x = 0 \text{ or } -2$$

		-2		0	
f'	$+$		$-$		$+$
	↗	Local max.	↘	Local Min	↗

$f(-2) = 4/e^2$

$f(0) = 0$

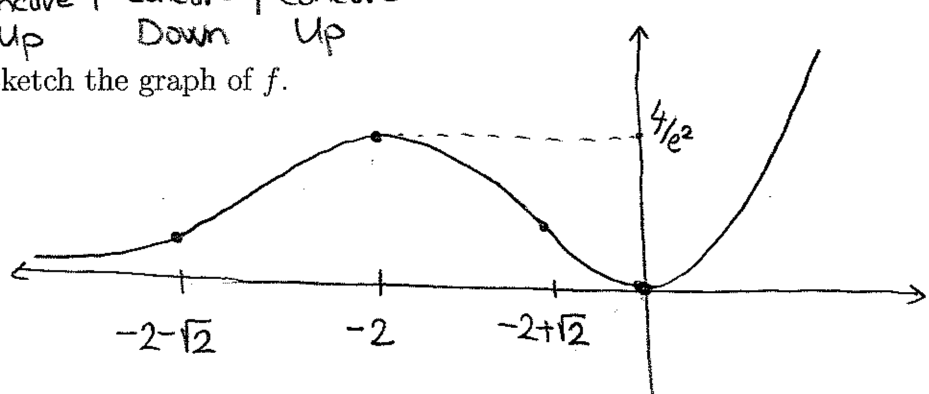
(D) Find intervals of concavity.

$$f''(x) = e^x(2+x) + x \cdot e^x(2+x) + x \cdot e^x = e^x(2+x+2x+x^2+x)$$

$$= e^x(x^2+4x+2) = 0 \Rightarrow x = -2-\sqrt{2} \text{ or } -2+\sqrt{2}$$

		$-2-\sqrt{2}$		$-2+\sqrt{2}$	
f''	$+$		$-$		$+$
	Concave Up		Concave Down		Concave Up

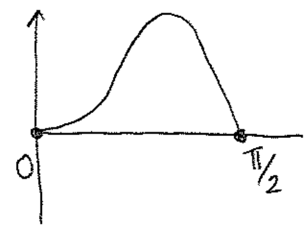
(E) Sketch the graph of f.



7. *(pts)* The following parts involve the curve $y = x^2 \sin(2x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

(A) Write, but **do NOT evaluate**, the integral which gives the **arclength** of $y = x^2 \sin(2x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

$$\int_0^{\pi/2} \sqrt{1 + \left(2x \cdot \sin(2x) + x^2 \cos(2x) \cdot 2\right)^2}\, dx$$



(B) Write, but **do NOT evaluate**, the integral which gives the **surface area** of the surface obtained by rotating this curve about x-axis.

$$\int 2\pi y\, ds = \int_0^{\pi/2} 2\pi x^2 \sin(2x) \cdot \sqrt{1 + \left(2x \sin(2x) + x^2 \cdot \cos(2x) \cdot 2\right)^2}\, dx$$

(C) Write, but **do NOT evaluate**, the integral which gives the **surface area** of the surface obtained by rotating this curve about x=-2 line.

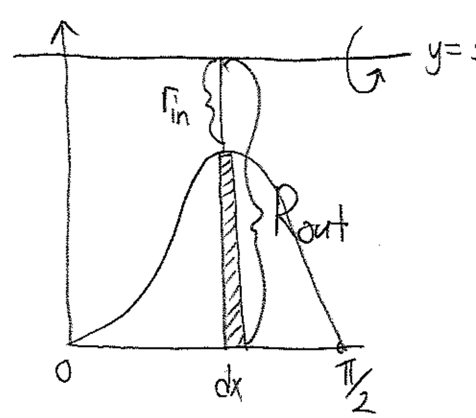
$$\int 2\pi (x+2)\, ds = \int_0^{\pi/2} 2\pi (x+2) \sqrt{1 + \left(2x \sin(2x) + x^2 \cos(2x) \cdot 2\right)^2}\, dx$$

(D) Write, but **do NOT evaluate,** the integral which gives the **volume** of the solids obtained by rotating the region under this curve from $x = 0$ to $x = \frac{\pi}{2}$ about y=5 line.

<u>Washer Method</u>: $R_{out} = 5$

$r_{in} = 5 - x^2 \sin(2x)$

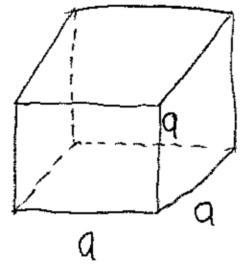
$$\int_0^{\pi/2} \left[\pi \cdot 25 - \pi\left(5 - x^2 \cdot \sin(2x)\right)^2\right] dx$$



8. *(pts)* The volume of a cube is growing at a constant rate of 7 $\frac{\text{cm}^3}{\text{s}}$. Find the rate of change of surface area at the instant that the side length is π cm.

$$V(t) = a^3(t) \qquad \frac{dV}{dt} = 7 \text{ cm}^3/s$$

$$S(t) = 6a^2(t) \qquad \left.\frac{dS}{dt}\right|_{a=\pi} = ?$$



$$7 = \frac{dV}{dt} = 3a^2(t) \cdot \frac{da}{dt}$$ when $a(t) = \pi$ we get $7 = 3 \cdot \pi^2 \cdot \frac{da}{dt}$

So, when $a(t) = \pi$, $\frac{da}{dt} = \frac{7}{3\pi^2}$ cm/s

At the same moment,

$$\left.\frac{dS}{dt}\right|_{a(t)=\pi} = 12 \cdot a(t) \cdot \frac{da}{dt} = 12 \cdot \pi \cdot \frac{7}{3\pi^2} = \frac{28}{\pi} \text{ cm}^2/s.$$

Bonus. A curve $y = f(x)$ (with $f(x) \geq 0$) is rotated around the line $y = -1$.

- Let $L(t)$ be the arc length of the curve $y = f(x)$ from $x = 0$ to $x = t$.
- Let $A(t)$ be the area of the surface of revolution when $y = f(x)$ is rotated around $y = -1$ (from $x = 0$ to $x = t$).

What is the ratio of change in surface area to change in arc length: $\frac{\frac{d}{dt}A(t)}{\frac{d}{dt}L(t)}$?

Surface Area $= A(t) = \int_0^t 2\pi (f(x)+1)\sqrt{1+(f'(x))^2}\, dx$

Arc Length $= L(t) = \int_0^t \sqrt{1+(f'(x))^2}\, dx$

$$\frac{\frac{dA}{dt}}{\frac{dL}{dt}} \overset{\text{F.T.C.}}{=} \frac{2\pi(f(t)+1)\cancel{\sqrt{1+(f'(t))^2}}}{\cancel{\sqrt{1+(f'(t))^2}}} = 2\pi(f(t)+1)$$