METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES
MIDTERM 2

Code : *MAT 120*	Last Name:
Acad.Year: *2013-2014*	Name :
Semester : *SPRING*	Student # : Solutions
Date : *03.05.2014*	Signature :
Time : *09:40*	4 QUESTIONS ON 4 PAGES
Duration : *90 min*	TOTAL 100 POINTS

1. (21)	2. (27)	3. (24)	4. (28)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(3×7=21pts)* Compute the following double integrals.
(You do not need to simplify your answers – e.g. "$2+3-\frac{1}{2}-\frac{1}{3}$" is fine.)

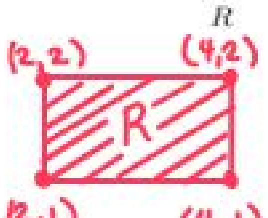
(a) $\displaystyle\int_1^3\int_{-1}^2 6x^2y + \frac{2x}{y}\,dx\,dy$

$$\int_{y=1}^{y=3}\left(\int_{x=-1}^{x=2} 6x^2y + \frac{2x}{y}\,dx\right)dy = \int_{y=1}^{y=3}\left(2x^3y + x^2/y\,\Big|_{x=-1}^{x=2}\right)dy = \int_{y=1}^{y=3}(16+2)y + \frac{(4-1)}{y}\,dy$$

$$= 18\cdot\tfrac12 y^2 + 3\ln|y|\,\Big|_{y=1}^{y=3}$$

$$= 9(9-1) + 3(\ln 3 - \ln 1) = \boxed{72 + 3\ln 3}$$

(b) $\displaystyle\iint_R \boxed{xy}\cos(\underline{1+xy^2})\,dA$ where R is the rectangle with corners $(2,2)$, $(2,1)$, $(4,1)$, $(4,2)$.



Substitute: $u = 1+xy^2$ → $du = y^2\,dx$
-or-
→ $du = \boxed{2xy}\,dy$ ← Best order

$$\int_{x=2}^{x=4}\int_{y=1}^{y=2} xy\cos(1+xy^2)\,dy\,dx = \int_{x=2}^{x=4}\tfrac12\sin(1+xy^2)\,\Big|_{y=1}^{y=2}\,dx$$

$$= \int_{x=2}^{x=4}\tfrac12\left(\sin(1+4x) - \sin(1+x)\right)dx$$

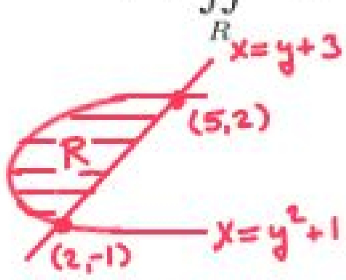
$$= \tfrac12\left(-\tfrac14\cos(1+4x) + \cos(1+x)\right)\Big|_{x=2}^{x=4}$$

$$= \boxed{-\tfrac18\cos 17 + \tfrac12\cos 5 + \tfrac18\cos 9 - \tfrac12\cos 3}$$

(c) $\displaystyle\iint_R 2xy\,dA$ where R is the region inside $\underline{y^2} = x-1$ and $y = x-3$

Must integrate dx first!

$y^2 = x-1$ → $x = y^2+1$
$y = x-3$ → $x = y+3$



Intersect at
$y^2+1 = y+3$
$y^2 - y - 2 = 0$
$(y-2)(y+1) = 0$
$y = -1, 2$

$$\int_{y=-1}^{y=2}\int_{x=y^2+1}^{x=y+3} 2xy\,dx\,dy$$

$$= \int_{y=-1}^{y=2} x^2y\,\Big|_{x=y^2+1}^{x=y+3}\,dy = \int_{y=-1}^{y=2}\left((y+3)^2 - (y^2+1)^2\right)y\,dy$$

$$= \int_{y=-1}^{y=2}\left(y^2+6y+9-y^4-2y^2-1\right)y\,dy = -\tfrac16 y^6 - \tfrac14 y^4 + \tfrac63 y^3 + \tfrac82 y^2\,\Big|_{y=-1}^{y=2}$$

$$= \boxed{-\tfrac{2^6}{6} - \tfrac{2^4}{4} + 2\cdot2^3 + 4\cdot2^2 + \tfrac16 + \tfrac14 + 2 - 4 = 15 + 3/4}$$

(simplification is not required)

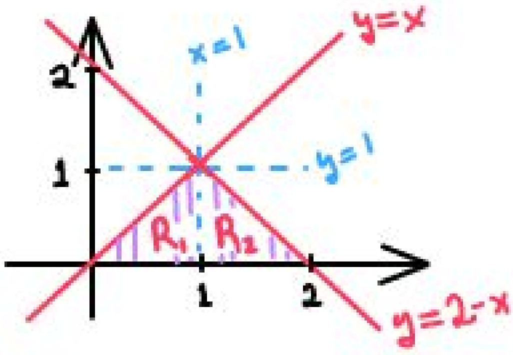
2. *(3×9=27pts)* Change the following integrals in the indicated manner, but DO NOT INTEGRATE.

(a) Reverse the order of integration.

$$\int_0^1 \int_y^{2-y} f(x,y)\,dx\,dy$$

The region is bounded by:

$\cancel{y=1}$ $y=0$ and $\left.\begin{array}{l} x=2-y \\ x=y \end{array}\right\}$ → Intersect at $2-y=y$, $2=2y$, $\underline{\underline{y=1}}$



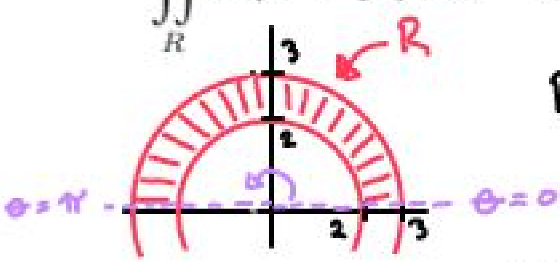
Reversing variables gives boundary

$y=2-x$
$y=x$ (and intersections) ⇒ $\underline{\text{Two}}$ integrals are needed!
$y=0$

$$\iint_{R_1} f\,dy\,dx + \iint_{R_2} f\,dy\,dx = \boxed{\int_{x=0}^{x=1}\int_{y=0}^{y=x} f\,dy\,dx + \int_{x=1}^{x=2}\int_{y=0}^{y=2-x} f\,dy\,dx}$$

(b) Change to polar coordinates.

$\displaystyle\iint_R \ln(x^2+y^2)\,dA$ where R is region given by $R = \{(x,y) : 4 \leqslant x^2+y^2 \leqslant 9,\ \text{and } y \geqslant 0\}$



R is $\begin{cases} r \text{ from } r=2 \text{ to } r=3 \\ \theta \text{ from } \theta=0 \text{ to } \theta=\pi \end{cases}$

$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \rightsquigarrow \ln(x^2+y^2) = \ln r^2$ $\qquad dA = r\,dr\,d\theta$

Integral is $\boxed{\displaystyle\int_{\theta=0}^{\theta=\pi}\int_{r=2}^{r=3} (\ln r^2)\,r\,dr\,d\theta}$

(c) Change coordinates $u = xy$ and $v = xy^3$.

$\displaystyle\iint_R xy^3\cos(xy)\,dx\,dy$ where R is the region in the first quadrant of xy-plane bounded by the curves $xy=4$, $xy=8$, $xy^3=5$ and $xy^3=15$.

$\underline{\text{Boundary}}$: $xy=4 \rightsquigarrow \boxed{u=4}$, $xy=8 \rightsquigarrow \boxed{u=8}$ $\qquad xy^3=5 \rightsquigarrow \boxed{v=5}$, $xy^3=15 \rightsquigarrow \boxed{v=15}$

$\underline{\text{Function}}$: $xy^3\cos(xy) \rightsquigarrow v\cos u$

$\underline{\text{Jacobian}}$: $\left.\begin{array}{l} u=xy \\ v=xy^3 \end{array}\right\}$ $du\,dv = \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = \left|\partial u/\partial x\,\partial v/\partial y - \partial u/\partial y\,\partial v/\partial x\right| dx\,dy$

$= \left|y\cdot 3xy^2 - x\cdot y^3\right| dx\,dy = \left|2xy^3\right| dx\,dy$

because $5 \le v \le 15$

So $dx\,dy = \dfrac{1}{|2xy^3|}\,du\,dv = \dfrac{1}{|2v|}\,du\,dv = \dfrac{1}{2v}\,du\,dv$

$\underline{\text{Integral}}$: $\displaystyle\int_{v=5}^{v=15}\int_{u=4}^{u=8} (v\cos u)\cdot\frac{1}{2v}\,du\,dv = \boxed{\int_{v=5}^{v=15}\int_{u=4}^{u=8} \frac{1}{2}\cos u\,du\,dv}$

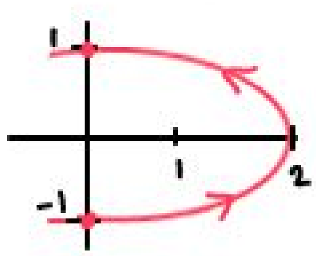
3. *(3×8=24pts)* Compute the following line integrals.

(a) $\int_C x^2 y\, ds$ where C is the line segment from $(0,1)$ to $(1,2)$

<u>Parametrize C</u>: $\underline{r}(t) = (0,1) + t\left((1,2) - (0,1)\right) = (t, 1+t)$ $\quad x(t) = t,\ y(t) = 1+t \quad ds = \sqrt{1^2 + 1^2}\, dt = \sqrt{2}\, dt$

<u>Integrate</u>:
$$\int_C x^2 y\, ds = \int_{t=0}^{t=1} (t)^2 \cdot (1+t) \cdot \sqrt{2}\, dt$$
$$= \sqrt{2} \int_{t=0}^{t=1} t^3 + t^2\, dt$$
$$= \sqrt{2}\left(\tfrac{1}{4}t^4 + \tfrac{1}{3}t^3 \Big|_{t=0}^{t=1}\right) = \sqrt{2}\left(\tfrac{1}{4} + \tfrac{1}{3}\right) = \boxed{\frac{7\sqrt{2}}{12}}$$

(b) $\int_C y\, dx - x\, dy$ where C is the right half of the ellipse, $x^2 + 4y^2 = 4$, moving counter-clockwise from $(0,-1)$ to $(0,1)$.



<u>Parametrize C</u>: $x^2 + 4y^2 = 4 \quad \begin{cases} x\text{-radius} = 2 & \rightsquigarrow x = 2\cos t \\ y\text{-radius} = 1 & \rightsquigarrow y = 1\sin t \end{cases}$

<u>Check</u>: $(2\cos t)^2 + 4(\sin t)^2 = 4\cos^2 t + 4\sin^2 t = 4$ ✓

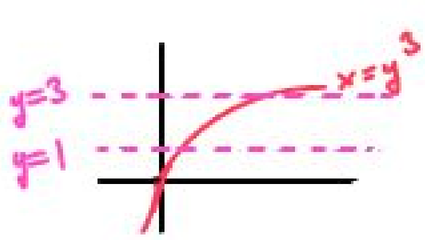
$x = 2\cos t \rightsquigarrow dx = -2\sin t\, dt$

$y = \sin t \rightsquigarrow dy = \cos t\, dt$

Right-hand side $\Rightarrow$ t from $-\pi/2$ to $\pi/2$

<u>Integrate</u>:
$$\int_C y\, dx - x\, dy = \int_{t=-\pi/2}^{t=\pi/2} (\sin t)(-2\sin t) - (2\cos t)(\cos t)\, dt$$
$$= \int_{t=-\pi/2}^{t=\pi/2} -2(\sin^2 t + \cos^2 t)\, dt = \int_{t=-\pi/2}^{t=\pi/2} -2\, dt = -2t \Big|_{t=-\pi/2}^{t=\pi/2} = \boxed{-2\pi}$$

(c) $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle y, x \rangle$ and C is the curve $x = y^3$ from $y = 1$ to $y = 3$.



<u>Parametrize C</u>: $x = y^3 \rightsquigarrow \begin{matrix} y = t \\ x = t^3 \end{matrix}$ from $t=1$ to $t=3$

$y = t \rightsquigarrow dy = 1 \cdot dt$

$x = t^3 \rightsquigarrow dx = 3t^2 \cdot dt$

<u>Integrate</u>:
$$\int_C \underline{F} \cdot d\underline{r} = \int_C y\, dx + x\, dy$$
$$= \int_{t=1}^{t=3} t \cdot 3t^2 + t^3\, dt$$
$$= \int_{t=1}^{t=3} 4t^3\, dt$$
$$= t^4 \Big|_{t=1}^{t=3} = 81 - 1 = \boxed{80}$$

4. *(3+3+8+8+6=28pts)* The following parts are about conservative/nonconservative vector fields and the Fundamental Theorem of Line Integrals.

(a) Is $\mathbf{F} = \langle \underbrace{ye^x + xye^x + 1}_{\partial/\partial y},\ \underbrace{xe^x + 2y}_{\partial/\partial x} \rangle$ conservative? (Show why or why not.)

$$e^x + xe^x \stackrel{\checkmark}{=} e^x + xe^x$$

$\boxed{\text{Conservative.}}$

(b) Is $\mathbf{F} = \underbrace{2xy\cos(x^2y)}_{\partial/\partial y}\mathbf{i} \underbrace{- x^2\cos(x^2y)}_{\partial/\partial x}\mathbf{j}$ conservative? (Show why or why not.)

$$2x\cos(x^2y) - 2x^3y\sin(x^2y) \neq -2x\cos(x^2y) + x^2 \cdot 2xy\sin(x^2y)$$

$\boxed{\text{Not conservative!}}$

(c) The vector field $\mathbf{F} = (2xy + 2)\mathbf{i} + (x^2 - 2y)\mathbf{j}$ is conservative. Note: $\partial/\partial y\,(2xy+2) = 2x$ so F is conservative.
Find a potential function ϕ with $\nabla\phi = \mathbf{F}$. $\partial/\partial x\,(x^2 - 2y) = 2x$

$$\begin{pmatrix}\text{x-terms}\\ \text{of } f\end{pmatrix} = \int 2xy + 2\, dx = \underbrace{x^2y + 2x}_{\partial/\partial y}$$

$$\begin{pmatrix}\text{Part of Q already}\\ \text{integrated}\end{pmatrix} = x^2$$

$$\begin{pmatrix}\text{y-terms}\\ \text{of } f\end{pmatrix} = \int \cancel{x^2} - 2y\, dy = -y^2$$

$$\boxed{f = x^2y + 2x - y^2}$$

Check: $\nabla f = \langle 2xy + 2,\ x^2 - 2y \rangle \stackrel{\checkmark}{=} F$

(d) Use the Fundamental Theorem of Line Integrals to calculate:

$$\int_C (2xy + 2)\, dx \ + (x^2 - 2y)\, dy$$

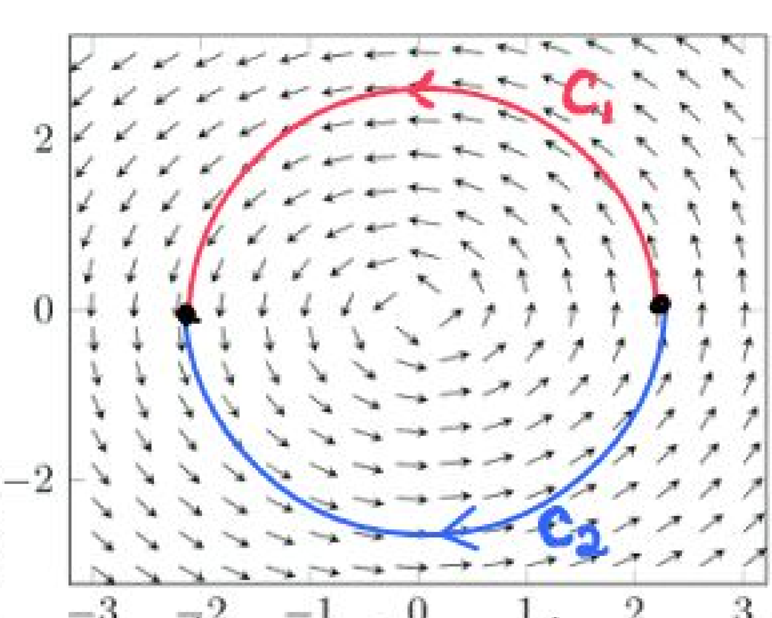
where C is $\left\{\mathbf{r}(t) = \langle \sin(3t) + 1,\ \cos(4t) - 1 \rangle \text{ for } t \text{ from } 0 \text{ to } \frac{\pi}{2}\right\}$

$$\text{start} = \underline{r}(0) = \langle \sin(0) + 1,\ \cos(0) - 1 \rangle = \langle 1, 0 \rangle$$

$$\text{end} = \underline{r}(\pi/2) = \langle \sin(3\pi/2) + 1,\ \cos(4\pi/2) - 1 \rangle = \langle 0, 0 \rangle$$

$$\int_C \underline{F} \cdot d\underline{r} = \phi \Big|_{\text{start}}^{\text{end}} = x^2y + 2x - y^2 \Big|_{x=1,\ y=0}^{x=0,\ y=0} = 0 - (2) = \boxed{-2}$$

(e) A vector field **F** is graphed to the right. By looking at the graph, state whether **F** is conservative or not conservative. Give reasons why or why not.
(Your answer **cannot** include guesses about what the functions P and Q are.)



F is not conservative.

Consider the two marked paths from (2,0) to (-2,0)

$\int_{C_1} \underline{F} \cdot d\underline{r} > 0$ but $\int_{C_2} \underline{F} \cdot d\underline{r} < 0$ Since these are not equal F cannot be conservative!!