METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES
MIDTERM 2

Code : *MAT 120*
Acad.Year: *2012-2013*
Semester : *Spring*
Date : *27.04.2013*
Time : *9:40*
Duration : *110 min*

Last Name:
Name : Student No.:
Department: Section:
Signature :

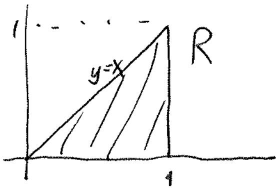
5 QUESTIONS ON 4 PAGES
TOTAL 100 POINTS

1. (20)	2. (20)	3. (20)	4. (20)	5. (20)		

1. *(6+6+8pts)* Compute the following integrals.

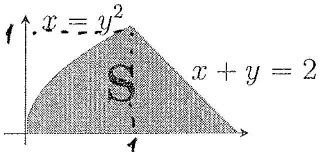
(A) $\iint_R e^{x^2}\,dA$ where R is the region bounded by $y = x$, $y = 0$ and $x = 1$.

$$\iint_R e^{x^2} dA = \int_0^1\int_0^x e^{x^2} dy\,dx = \int_0^1 e^{x^2} y\Big|_0^x dx = \int_0 x e^{x^2} dx$$



$$= \frac{1}{2}\int_0^1 2xe^{x^2} dx = \frac{e^{x^2}}{2}\Big|_0^1 = \frac{e-1}{2}.$$

(C) $\iint_S 2xy\,dA$ where S is the region bounded by $\begin{cases} x = y^2, \\ x + y = 2, \\ y = 0. \end{cases}$



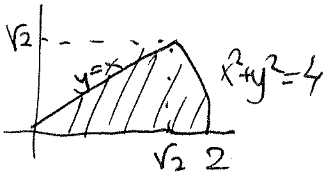
$$= \int_0^1\int_{y^2}^{2-y} 2xy\,dx\,dy = \int_0^1 yx^2\Big|_{y^2}^{2-y} dy = \int_0^1 y(2-y)^2 - y^5\,dy$$

$$= \int_0^1 4y - 4y^2 + y^3 - y^5\,dy = 2y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4 - \frac{1}{6}y^6\Big|_0^1 = 2 - \frac{4}{3} + \frac{1}{4} - \frac{1}{6}$$

$$= 3/4$$

Move to page 2.

(B) $\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} e^{x^2+y^2}\,dx\,dy$ $= \iint_D e^{x^2+y^2} dA$ whre D is



$$= \int_0^{\pi/4}\int_0^2 e^{r^2} r\,dr\,d\theta = \frac{\pi}{4}\left(\frac{e^4-1}{2}\right) = \frac{\pi}{8}(e^4-1)$$

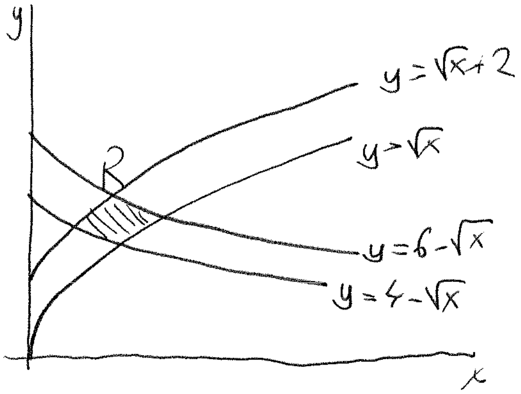
use polar coordinates

2. *(20pts)* Change variables to (u, v) where $u = \sqrt{x}$, $v = y - \sqrt{x}$ and integrate.

$$\iint\limits_R \frac{e^{y-\sqrt{x}}}{\sqrt{x}}\, dA.$$ R is the region inside $y = \sqrt{x}$, $y = \sqrt{x} + 2$, $y = 4 - \sqrt{x}$, $y = 6 - \sqrt{x}$.

$u = \sqrt{x} \quad \frac{4-v}{2} \leq u \leq \frac{6-v}{2}$ (because $y + \sqrt{x} = 2u + v$ and $4 \leq y + \sqrt{x} \leq 6$)

$v = y - \sqrt{x} \quad 0 \leq v \leq 2$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} \frac{1}{2\sqrt{x}} & 0 \\ \frac{-1}{2\sqrt{x}} & 1 \end{vmatrix}} = 2\sqrt{x} = \boxed{2u}$$

$$\iint\limits_R \frac{e^{y-\sqrt{x}}}{\sqrt{x}} dA = \int_0^2 \int_{\frac{4-v}{2}}^{\frac{6-v}{2}} \frac{e^v}{u} 2u \, du\, dv = \int_0^2 \int_{\frac{4-v}{2}}^{\frac{6-v}{2}} 2e^v du\, dv$$

$$= \int_0^2 2e^v \, u\Big|_{\frac{4-v}{2}}^{\frac{6-v}{2}} dv = \int_0^2 \cancel{2}e^v \frac{6-\cancel{v}-4+\cancel{v}}{\cancel{2}} dv = \int_0^2 2e^v dv$$

$$= 2e^v\Big|_0^2 = 2(e^2-1).$$

3. *(6+6+8pts)* Compute the following line integrals.

(A) $\int_C 1 + 9x \, ds$ $\quad$ C is $\mathbf{r}(t) = \langle t,\ 2t^{\frac{3}{2}} \rangle$ for $0 \leqslant t \leqslant 1$. $\quad$ $r'(t) = < 1,\ 3t^{1/2} >$

$$= \int_C (1+9x(t))|r'(t)|\,dt = \int_0^1 (1+9t)\sqrt{1^2+(3t^{1/2})^2}\,dt = \int_0^1 (1+9t)^{3/2}dt$$

$$= \frac{1}{9}\int_0^1 (1+9t)^{3/2}\,9dt = \frac{1}{9}\int_1^{10} u^{3/2}du = \frac{1}{9}\,\frac{2u^{5/2}}{5}\Big|_1^{10} = \frac{2}{45}\left(10^{5/2}-1\right)$$

$(u = 1+9t)$

(B) $\int_C (1+xy)\,dx + (y-x)\,dy$ $\quad$ C is $\mathbf{r}(t) = \langle t^2,\ t^3 \rangle$ for $-1 \leqslant t \leqslant 1$. $\quad$ $r' = (x', y') = (2t, 3t^2)$

$$= \int_C (1+xy)dx + \int_C (y-x)dy = \int_{-1}^1 (1+t^5)\,2t\,dt + \int_{-1}^1 (t^3-t^2)\,3t^2dt$$

$$= \int_{-1}^1 2t+2t^6+3t^5-3t^4\,dt = t^2+\frac{2t^7}{7}+\frac{t^6}{2}-\frac{3t^5}{5}\Big|_{-1}^1 = 1+\frac{2}{7}+\frac{1}{2}-\frac{3}{5}$$

$$-1+\frac{2}{7}-\frac{1}{2}-\frac{3}{5}$$

$$= \frac{22}{35}$$

(C) $\int_C \underbrace{\left(y^2\cos(xy^2)+y\right)}_{=P} dx + \underbrace{\left(2xy\cos(xy^2)+x+y\right)}_{=Q} dy$ $\quad$ C is the line $(1,2)$ to $(4,1)$.

$P_y = 2y\cos(xy^2) - 2y^3x\sin(xy^2)+1 = Q_x \Rightarrow F=(P,Q)$ is conservative on $\mathbb{R}^2$ which is simply connected.

$\Rightarrow F = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$

Find f such an: $\quad f = \sin(xy^2)+xy+\frac{y^2}{2}+C$

FTLI $\Rightarrow$ $\int_C F\,dr = f(4,1)-f(1,2) = \cancel{\sin 4}+\cancel{4}+\frac{1}{2}+\cancel{c}-\cancel{\sin 4}-\cancel{2}-\cancel{2}-\cancel{c}$

$$= \frac{1}{2}.$$

4. *(20pts)* **Use the method of Lagrange multipliers** to find the minimum and maximum values of $f(x,y) = e^{x+4y}$ on $x^2 + y^2 = 5$.

$g(x,y) = x^2+y^2-5$

the constraint is $g(x,y)=0$

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \\ g &= 0 \end{aligned} \Rightarrow \begin{aligned} e^{x+4y} - 2x\lambda &= 0 \\ 4e^{x+4y} - 2y\lambda &= 0 \\ x^2+y^2 &= 5 \end{aligned} \Rightarrow \begin{aligned} \lambda &= \frac{e^{x+4y}}{x} \\ \lambda &= \frac{4e^{x+4y}}{y} \end{aligned} \Rightarrow y = 4x$$

$x \neq 0$
$y \neq 0$
$\lambda \neq 0$

$$x^2+y^2=5 \Rightarrow x^2+16x^2=5 \Rightarrow x = \pm\sqrt{5/17}$$
$$y = \pm 4\sqrt{5/17}$$

$f\left(-\sqrt{5/17}, -4\sqrt{5/17}\right) = e^{-15\sqrt{5/17}}$ min value

$f\left(\sqrt{5/17}, 4\sqrt{5/17}\right) = e^{15\sqrt{5/17}}$ max value

Keep in mind that $x^2+y^2=5$ is bounded and closed so f has min/max on it.

5. *(20pts)* Give the limits of integration for $\iiint_D dV$ as an integral $\iiint dx\,dy\,dz$ and $\iiint dy\,dz\,dx$ if D is the region inside of $z = 4 - y^2$, $x + z = 4$, $x = 0$, $z = 0$. **(Do NOT integrate.)**
(For 8pts bonus, graph the region and give $\iiint dz\,dx\,dy$. You may need extra paper.)

$$\iiint_D dV = \int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} \int_0^{4-z} dx\,dy\,dz$$

$$= \int_0^4 \int_0^{4-z} \int_{-\sqrt{4-z}}^{\sqrt{4-z}} dy\,dz\,dx$$