METU - NCC

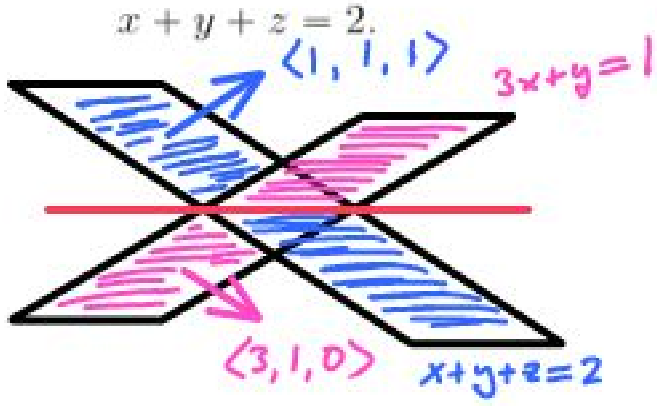
CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *FALL* Date : *02.11.2013* Time : *13:40* Duration : *110 min*	Last Name: Name : Student # : Signature : 5 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (20)	2. (20)	3. (20)	4. (20)	5. (20)				

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(2×6+8=20 pts)* The following parts are about vectors, lines, and planes in 3D.

(A) (i) Find a parametric equation for the intersection line of the two planes $3x + y = 1$ and $x + y + z = 2$.



For the line of intersection we need

- point on intersection:

$$\begin{aligned} 3x+y &= 1 \\ x+y+z &= 2 \\ x &= 0 \end{aligned} \quad \rightarrow y=1 \rightarrow z=1$$

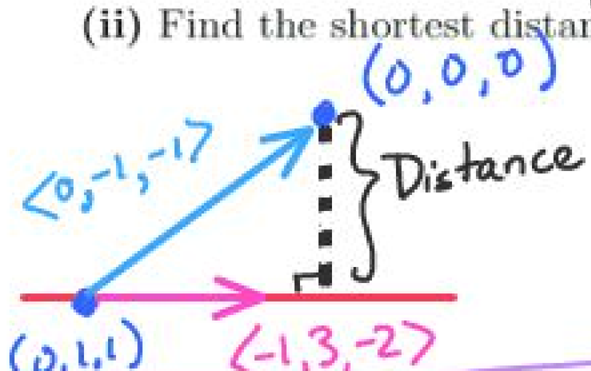
(0, 1, 1)

- direction of intersection:

$\langle 1,1,1\rangle \times \langle 3,1,0\rangle = \langle -1,3,-2\rangle$

$$\boxed{\text{Intersection: } \underline{r}(t) = \langle -1,3,-2\rangle t + \langle 0,1,1\rangle}$$

(ii) Find the shortest distance from the point (0, 0, 0) to the line found in part **(i)**.



$$\text{Distance} = \left| \text{Proj}^{\perp}_{\langle -1,3,-2\rangle} \langle 0,-1,-1\rangle \right|$$

$$= \left| \langle 0,-1,-1\rangle - \frac{\langle -1,3,-2\rangle \bullet \langle 0,-1,-1\rangle}{\langle -1,3,-2\rangle \bullet \langle -1,3,-2\rangle} \langle -1,3,-2\rangle \right|$$

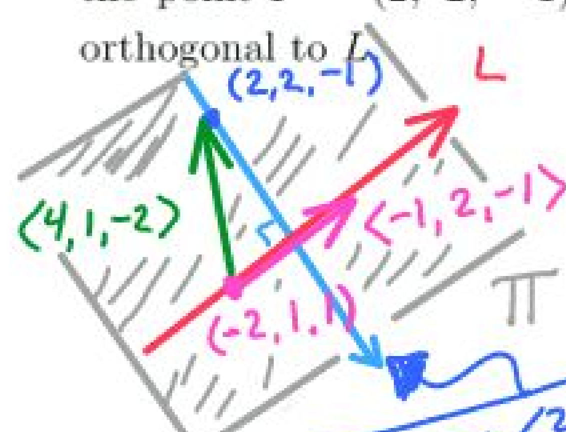
$$= \left| \langle 0,-1,-1\rangle + \tfrac{1}{14} \langle -1,3,-2\rangle \right|$$

$$= \tfrac{1}{14} \left| \langle 0,-14,-14\rangle + \langle -1,3,-2\rangle \right| = \tfrac{1}{14}\sqrt{378} = \boxed{\frac{3\sqrt{3}}{\sqrt{14}}}$$

This could also be computed as:

$$\frac{|\langle 0,-1,-1\rangle \times \langle -1,3,-2\rangle|}{|\langle -1,3,-2\rangle|}$$

(B) Let Π be the plane containing the line L given by $\mathbf{r}(t) = \langle -1,\ 2,\ -1\rangle t + \langle -2,\ 1,\ 1\rangle$ and the point $P = (2,\ 2,\ -1)$. Write the equation for the line in Π which goes through P and is orthogonal to L.



$$\boxed{s(t) = \langle -4,-1,2\rangle t + \langle 2,2,-1\rangle}$$

The line L has $\begin{cases} \text{point } (-2,1,1) \\ \text{direction } \langle -1,2,-1\rangle \end{cases}$

The plane Π has normal direction

$$\underline{n} = \langle -2,1,1\rangle \times \langle 4,1,-2\rangle = \langle -3,0,-6\rangle \sim \langle 1,0,2\rangle$$

The orthogonal line has direction

$$\underset{(\text{in plane } \Pi)}{\underline{v} = \langle 1,0,2\rangle} \times \underset{(\perp \text{ to } L)}{\langle -1,2,-1\rangle} = \langle -4,-1,2\rangle$$

2. *(4×4+4=20 pts)* The following problems are about quadric surfaces.

(A) Name the traces (e.g. hyperbola, parabola, etc) and surfaces below.

(i) $x^2 + 3y^2 - z - 1 = 0$

x-trace: parabola $3y^2 - z = k$ y-trace: parabola $x^2 - z = k$ z-trace: ellipse $x^2 + 3y^2 = k$

Name of surface: elliptic paraboloid

(ii) $2x^2 - 7y^2 + z^2 + 1 = 0$

x-trace: hyperboloid $-7y^2 + z^2 = k$ y-trace: ellipse $2x^2 + z^2 = k$ z-trace: hyperboloid $2x^2 - 7y^2 = k$

Name of surface: hyperboloid of <u>two</u> sheets

$7y^2 = 2x^2 + z^2 \underline{+1}$

(iii) $-2x^2 - 3y^2 + z + 1 = 0$

x-trace: parabola $-3y^2 + z = k$ y-trace: parabola $-2x^2 + z = k$ z-trace: ellipse $-2x^2 - 3y^2 = k$

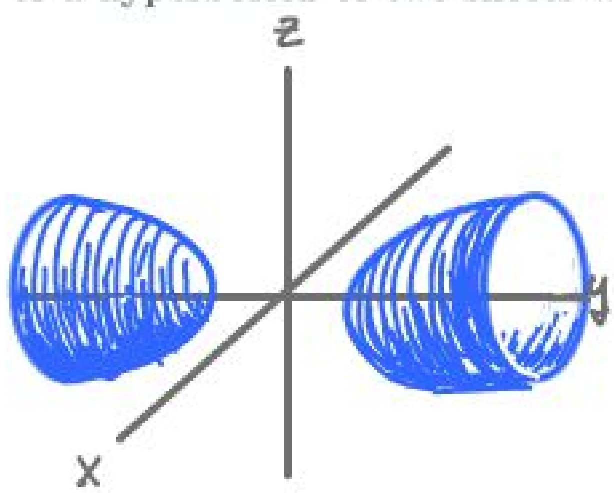
Name of surface: elliptic paraboloid

(iv) $6x^2 + 2y^2 + 4z^2 - 1 = 0$

x-trace: ellipse $2y^2 + 4z^2 = k$ y-trace: ellipse $6x^2 + 4z^2 = k$ z-trace: ellipse $6x^2 + 2y^2 = k$

Name of surface: ellipsoid

(B) Draw the graph of a hyperboloid of two-sheets which is symmetric around the y-axis (the y-traces are circles).



3. *(8+3×4=20 pts)* The following problems are about vector functions.

(A) Suppose that $\mathbf{r}(0) = \langle 1, 2, -1\rangle$ and $\mathbf{r}'(t) = \langle 2t+1, 1, 1-3t^2\rangle$. Find $\mathbf{r}(2)$.

$$\underline{r}(2) = \int_0^2 \underline{r}'(t)\,dt + \underline{r}(0) \qquad \text{(Net change theorem)}$$

$$= \left\langle \int_0^2 2t+1\,dt,\ \int_0^2 1\,dt,\ \int_0^2 1-3t^2\,dt \right\rangle + \langle 1,2,-1\rangle$$

$$= \left\langle t^2+t\Big|_0^2,\ t\Big|_0^2,\ t-t^3\Big|_0^2 \right\rangle + \langle 1,2,-1\rangle$$

$$= \langle 6, 2, -6\rangle + \langle 1, 2, -1\rangle = \boxed{\langle 7, 4, -7\rangle}$$

(B) The following parts use the vector functions $\mathbf{r}(t) = \langle \cos(t),\ \sin(t)\rangle$ and $\mathbf{s}(t) = \langle \cos(t^2),\ \sin(t^2)\rangle$ both of which trace out the circle $x^2+y^2=1$. Note that $\mathbf{r}(\pi/2) = \mathbf{s}(\sqrt{\pi/2}) = (0,1)$.

(i) Compute $\mathbf{r}'(\pi/2)$ and $\mathbf{s}'(\sqrt{\pi/2})$.

$$\underline{r}(t) = \langle \cos t, \sin t\rangle$$

$$\underline{r}'(t) = \langle -\sin t, \cos t\rangle$$

$$\underline{r}'(\pi/2) = \langle -\sin \pi/2, \cos \pi/2\rangle = \boxed{\langle -1, 0\rangle}$$

$$\underline{s}(t) = \langle \cos t^2, \sin t^2\rangle$$

$$\underline{s}'(t) = \langle -2t\sin t^2, 2t\cos t^2\rangle$$

$$\underline{s}'(\sqrt{\pi/2}) = \langle -2\sqrt{\pi/2}\sin \pi/2,\ 2\sqrt{\pi/2}\cos \pi/2\rangle = \boxed{\langle -2\sqrt{\pi/2}, 0\rangle}$$

(ii) Compute $\mathbf{r}''(\pi/2)$ and $\mathbf{s}''(\sqrt{\pi/2})$.

$$\underline{r}'(t) = \langle -\sin t, \cos t\rangle$$

$$\underline{r}''(t) = \langle -\cos t, -\sin t\rangle$$

$$\underline{r}''(\pi/2) = \langle -\cos \pi/2, -\sin \pi/2\rangle = \boxed{\langle 0, -1\rangle}$$

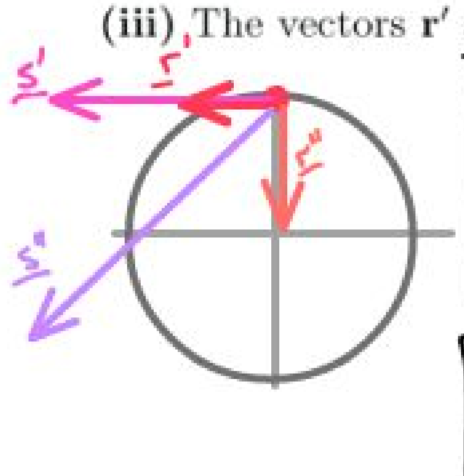
$$\underline{s}'(t) = \langle -2t\sin t^2, 2t\cos t^2\rangle$$

$$\underline{s}''(t) = \langle -2\sin t^2 - 4t^2\cos t^2,\ 2\cos t^2 - 4t^2\sin t^2\rangle$$

$$\underline{s}''(\sqrt{\pi/2}) = \langle -2\sin \pi/2 - 4(\pi/2)\cos \pi/2,\ 2\cos \pi/2 - 4(\pi/2)\sin \pi/2\rangle$$

$$= \boxed{\langle -2, -2\pi\rangle}$$

(iii) The vectors $\mathbf{r}'$ and $\mathbf{s}'$ point in the same direction, but $\mathbf{r}''$ and $\mathbf{s}''$ don't. Why not?



$\underline{r}'$ and $\underline{s}'$ point in the same direction because they both point in the direction of motion of a particle at position (0,1) orbiting in a circle.
(The length $|\underline{s}'|$ is bigger than $|\underline{r}'|$ because it moves faster)

$\underline{r}''$ and $\underline{s}''$ point in different directions because $\underline{r}$ moves at constant speed $= |r'| = \sqrt{\sin^2 t + \cos^2 t} = 1$ so the accel. $\underline{r}''$ is only making it turn ($\underline{r}'' \perp \underline{r}'$). BUT $\underline{s}$ is speeding up, so it must have some accel. in the direction of motion as well as $\perp \underline{s}'$

4. *(6+6+8=20 pts)* The following parts are about functions of several variables and their limits.

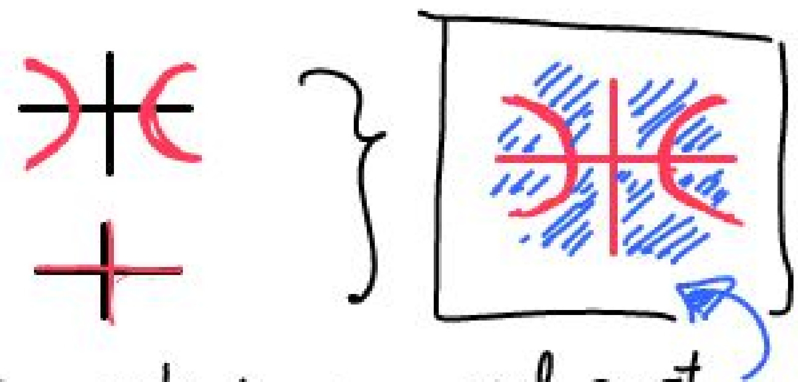
(A) Find and sketch the domain of the function

$$f(x,y) = \frac{\ln|xy|}{x^2 - y^2 - 1}.$$

Denominator $= 0$ if $x^2 - y^2 - 1 = 0$

$y^2 = x^2 - 1$

Numerator is undefined if $xy = 0$

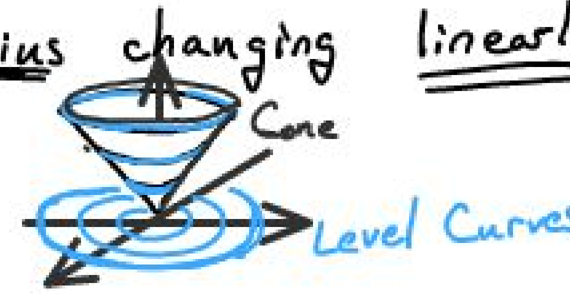


Domain is stuff which is <u>not</u> in any red part.

(B) Write a two variable function whose level curves are equally spaced circles.

If the level curves are equally spaced circles then $z = f(x,y)$ is <u>circles</u> w/ <u>radius</u> changing <u>linearly</u> w/ z

<u>EX</u> $z^2 = x^2 + y^2 \rightsquigarrow \boxed{z = \sqrt{x^2+y^2} \quad \text{Cone}}$



(C) Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^k}{x^2+y^2}$. Show that the limit does not exist if $k \leqslant 2$ and that it exists if $k > 2$.

Suppose $\underline{k>2}$. Note $\dfrac{x^k}{x^2+y^2} = x^{k-2} \cdot \dfrac{x^2}{x^2+y^2} \leq |x^{k-2}| \cdot 1$ $\left(\text{as long as } (x,y) \neq (0,0) \quad \dfrac{x^2}{x^2+y^2} \leq 1\right)$

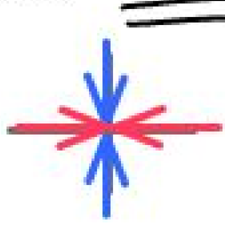
Since $k>2$, $k-2>0$ so $\displaystyle\lim_{x\to 0} x^{k-2} = 0$.

Apply the Squeeze Thm:

$$-|x^{k-2}| \leq \frac{x^k}{x^2+y^2} \leq |x^{k-2}|$$

and $\displaystyle\lim_{(x,y)\to(0,0)} x^{k-2} = 0$ so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^k}{x^2+y^2} = 0$ also.

Suppose $\underline{k=2}$. Approaching $(0,0)$ along different lines gives



$\left.\begin{array}{l} x=0 \\ y\to 0 \end{array}\right\} \displaystyle\lim_{y\to 0} \frac{0^2}{0^2+y^2} = 0$

These are <u>not</u> equal so the limit does <u>not</u> exist!

$\left.\begin{array}{l} x\to 0 \\ y=0 \end{array}\right\} \displaystyle\lim_{x\to 0} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

Suppose $\underline{k<2}$. Approaching $(0,0)$ along different lines gives

$\left.\begin{array}{l} x=0 \\ y\to 0 \end{array}\right\} \displaystyle\lim_{y\to 0} \frac{0^k}{0^2+y^2} = 0$

<u>Note:</u> $k<2$ so $k-2<0$

These are <u>not</u> equal so limit does <u>not</u> exist!

$\left.\begin{array}{l} x\to 0 \\ y=0 \end{array}\right\} \displaystyle\lim_{x\to 0} \frac{x^k}{x^2-0^2} = \lim_{x\to 0} x^{k-2} = \pm\infty$

5. *(3×4+8=20 pts)* Compute the indicated partial derivatives.

(A) $\frac{\partial}{\partial x}\left(y\sin(xy^2)\right)$

$$\frac{\partial}{\partial x}\left(y\sin(xy^2)\right) = y\cos(xy^2)\cdot\frac{\partial}{\partial x}(xy^2)$$
$$= y\cos(xy^2)\cdot y^2$$
$$= \boxed{y^3\cos(xy^2)}$$

(B) $\frac{\partial}{\partial y}\left(y\sin(xy^2)\right)$

$$\frac{\partial}{\partial y}\left(y\sin(xy^2)\right) = \sin(xy^2) + y\cos(xy^2)\cdot\frac{\partial}{\partial y}(xy^2)$$
$$= \boxed{\sin(xy^2) + 2xy^2\cos(xy^2)}$$

(C) $\frac{\partial^2}{\partial x\partial y}\left(2^{\tan(\ln x)} + \sec(\sqrt{y}\,e^y) + xy\right)$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(2^{\tan(\ln x)}\right) + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\sec(\sqrt{y}\,e^y)\right) + \frac{\partial}{\partial x}\frac{\partial}{\partial y}(xy)$$

$$= 0 + 0 + 1 = \boxed{1}$$

(D) Find an equation of the tangent plane to the surface $z = x^3 + xy + y^3$ at the point $(1, 2, 11)$

$f(x,y) = x^3 + xy + y^3$ $\qquad$ $f(1,2) = 1+2+8 = 11$ Duh.

$f_x(x,y) = 3x^2 + y$ $\qquad$ $f_x(1,2) = 3+2 = 5$

$f_y(x,y) = x + 3y^2$ $\qquad$ $f_y(1,2) = 1+12 = 13$

Tangent plane:

$$z = 5(x-1) + 13(y-2) + 11$$

equivalently:

$$z = 5x + 13y - 20$$

or

$$5x + 13y - z = 20$$