METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL	
Code : *MAT 119* Acad.Year: *2012-2013* Semester : *SUMMER* Date : *04.08.2013* Time : *16:00* Duration : *150 minutes*	Last Name: Name : KEY Department: Signature: Student No.: Section:
	6 QUESTIONS ON 8 PAGES TOTAL 100 POINTS

1. (15)	2. (30)	3. (15)	4. (20)	5. (10)	6. (10)	Bonus		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(5+5+5=15pts)* Find the following limits.

(a) $\lim_{x\to\infty} \frac{x\tan^{-1}x}{\ln\sqrt{x^2+1}} \overset{L'HR}{=} \lim_{x\to\infty} \frac{1\cdot\tan^{-1}x + \frac{x}{1+x^2}}{\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+1}} = \infty$

(b) $\lim_{x\to 0^+} (\sin x)^{\tan x} = e^{\lim_{x\to 0^+} \ln(\sin x)^{\tan x}}$

$= e^{\lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x}}$

$\overset{L'HR}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x}}$

$= e^{\lim_{x\to 0^+} -\sin x\cos x}$

$= e^0 = 1.$

(b) $\lim_{x\to 0^+} [\frac{1}{x} - \frac{1}{\ln(x+1)}] = \lim_{x\to 0^+}\left[\frac{\ln(x+1)-x}{x\ln(x+1)}\right] \overset{L'HR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x+1}-1}{\ln(x+1)+\frac{x}{x+1}}$

$= \lim_{x\to 0^+} \frac{-x}{(x+1)\ln(x+1)+x}$

$\overset{L'HR}{=} \lim_{x\to 0^+} \frac{-1}{\ln(x+1)+\frac{x+1}{x+1}+1}$

$= -\frac{1}{2}.$

2. (5+5+5+5+5+5=30pts) Compute the following integrals.

(a) $\int \cos x \cos^3(\sin x)dx \underset{\substack{\sin x = u \\ \cos x\, dx = du}}{=} \int \cos^3 u\, du = \int (1-\sin^2 u)\cos u\, du \underset{\substack{v=\sin u \\ dv = \cos u\, du}}{=} \int (1-v^2)dv$

$= v - \frac{v^3}{3} + C = \sin u - \frac{\sin^3 u}{3} + C$

$= \sin(\sin x) - \frac{\sin^3(\sin x)}{3} + C$

(b) $\int \frac{1}{e^x\sqrt{1-e^{-2x}}}dx \underset{\substack{e^{-x}=u \\ -e^{-x}dx = du}}{=} \int -\frac{du}{\sqrt{1-u^2}} = \cos^{-1} u + C$

$= \cos^{-1}(e^{-x}) + C$

(c) $\int \frac{\ln(x+1)}{x^2}dx = -\frac{1}{x}\cdot\ln(x+1) - \int -\frac{1}{x}\cdot\frac{dx}{x+1}$

$\ln(x+1) = u \Rightarrow du = \frac{dx}{x+1}$

$\frac{dx}{x^2} = dv \Rightarrow v = -\frac{1}{x}$

$\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$

$= -\frac{\ln(x+1)}{x} + \ln x - \ln(x+1) + C$

4. *(5+5+5+5=20pts)* DO NOT COMPUTE THE VALUE OF THE INTEGRALS

(a) Write the arclength of $y = \ln x$ on $[1, e]$ as an integral.

$$L = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$$

(b) Write the surface area of the solid obtained by rotating the region between $y = \ln x$ and x-axis on $[1, e]$ around the y-axis as an integral.

$$S.A = \int_1^e 2\pi x \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) Write the volume of the solid obtained by rotating the region between $y = \ln x$ and $y = 1$ on $[1, e]$ around the $y = -1$ as an integral.

$$V = \int_1^e \pi \left((1+1)^2 - (\ln x + 1)^2 \right) dx$$

(d) Write the volume of the solid obtained by rotating the region between $y = \ln x$ and $y = 1$ on $[1, e]$ around the $x = e$ as an integral.

$$V = \int_1^e 2\pi (e-x)(1 - \ln x)\, dx$$

5. *(1+1+1+2+3+2=10pts)* Let $f(x) = \sqrt{x^2 - 2x - 3}$. Its derivatives are given as follows
$f'(x) = \dfrac{x-1}{\sqrt{x^2 - 2x - 3}}$, $f''(x) = -\dfrac{4}{(x^2 - 2x - 3)^{\frac{3}{2}}}$

(a) Domain of $f(x)$: $\mathbb{R} - (-1, 3)$

$x^2 - 2x - 3 = (x-3)(x+1)$

(b) Intercepts: No y intercept. $x = -1$ and $x = +3$ are x intercepts.

(c) Symmetry/Periodicity: $f(-x) = \sqrt{(-x)^2 - 2(-x) - 3} \neq f(x)$ or $f(-x)$

It has no symmetry.

(d) Asypmtotes: $\lim\limits_{x \to \infty} \sqrt{x^2 - 2x - 3} - mx - b = \lim\limits_{x \to \mp\infty} \dfrac{(\sqrt{x^2 - 2x - 3} - mx - b)(\sqrt{x^2 - 2x - 3} + mx + b)}{\sqrt{x^2 - 2x - 3} + mx + b}$

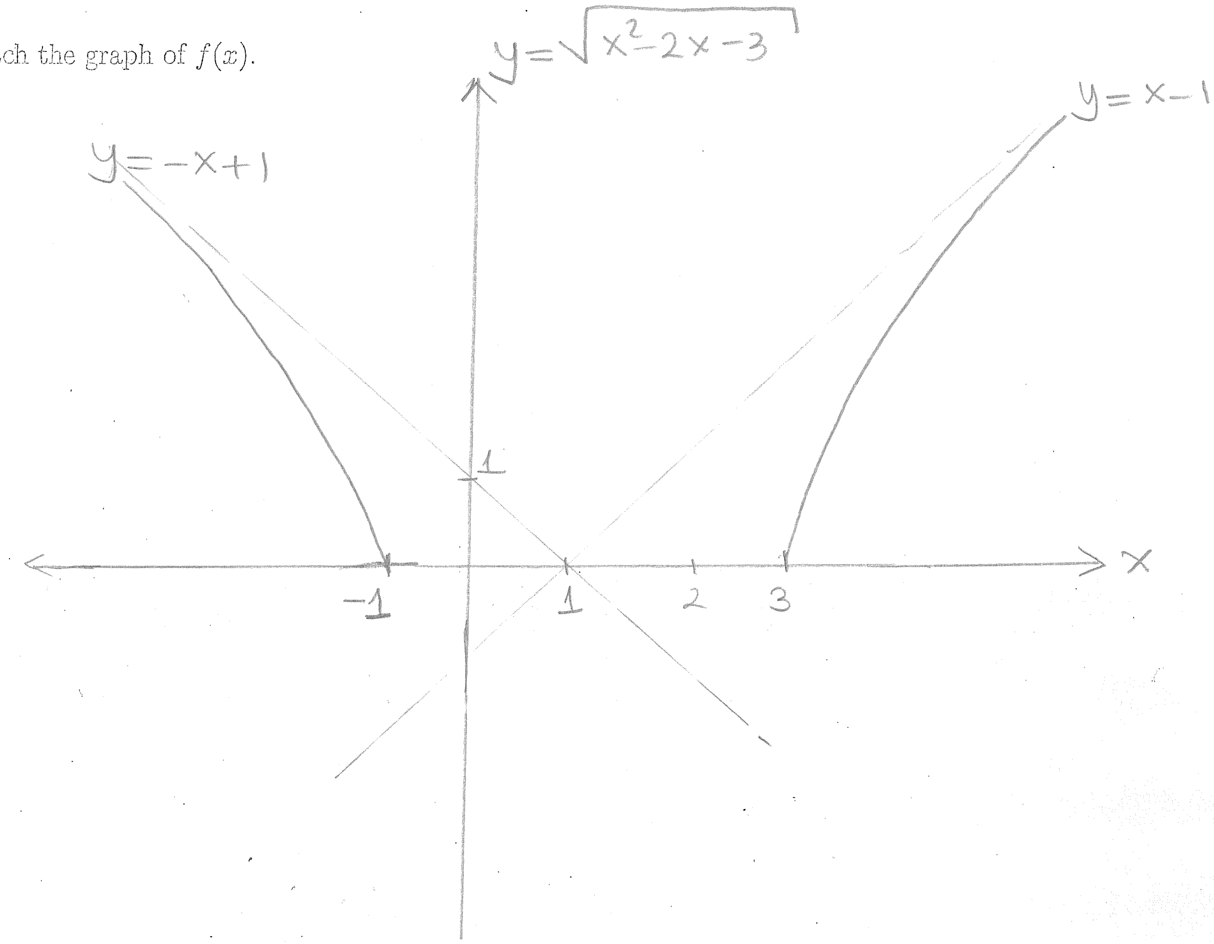
$= \lim\limits_{x \to \mp\infty} \dfrac{(1-m^2)x^2 - (2+2mb)x - b^2}{\sqrt{x^2 - 2x - 3} + mx + b} = 0$

So, $\left.\begin{array}{l} 1 - m^2 = 0 \Rightarrow m = \mp 1 \\ 2 + 2mb = 0 \Rightarrow b = -\frac{1}{m} \end{array}\right\} \Rightarrow$ $y = x - 1$ and $y = -x + 1$ are slant asymptotes.

(e) Intervals of increasing/decreasing and concave up/down

		−1		1		3	
$f(x)$	+						+
$f'(x)$	−						+
$f''(x)$	−						−

(f) Sketch the graph of $f(x)$.



3. *(5+5+5=15pts)* Evaluate the integral or show that it is divergent

(a) $\int_0^4 \frac{\ln x}{\sqrt{x}} dx = \lim_{t\to 0^+} \int_t^4 \frac{\ln x}{\sqrt{x}} dx = \lim_{t\to 0^+} \left[2\sqrt{x}\ln x - \int \frac{2\sqrt{x}}{x} dx \right]\Big|_t^4$

$= \lim_{t\to 0^+} \left[2\sqrt{x}\ln x - 4\sqrt{x} \right]\Big|_t^4$

$= \lim_{t\to 0^+} \left[8(\ln 2 - 1) - 2\sqrt{t}\ln t - 4\sqrt{t} \right]$

$= \lim_{t\to 0^+} \left[8(\ln 2 - 1) - \frac{\ln t}{\frac{1}{2\sqrt{t}}} - 4\sqrt{t} \right]$

$\overset{LHR}{=} \lim_{t\to 0^+} \left[8(\ln 2 - 1) - \frac{\frac{1}{t}}{\frac{1}{3t\sqrt{t}}} - 4\sqrt{t} \right]$

$= 8(\ln 2 - 1)$

(b) $\int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1} x}{x^2} dx = \lim_{t\to\infty} \left[-\frac{1}{x}\tan^{-1} x - \int \frac{dx}{x(1+x^2)} \right]\Big|_1^t$

$= \lim_{t\to\infty} \left[-\frac{\tan^{-1} x}{x} - \int \frac{x}{x^2+1} - \frac{1}{x} dx \right]\Big|_1^t = \lim_{t\to\infty} -\frac{\tan^{-1} x}{x} - \frac{1}{2}\ln|x^2+1| + \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \left[\left(-\frac{\tan^{-1} t}{t} + \ln\left|\frac{t}{\sqrt{t^2+1}}\right| \right) - \left(-\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} \right) \right]$

$= \frac{\pi}{4} + \ln\sqrt{2}$

(c) $\int_{-1}^{+1} \frac{1}{x^2 - 2x} dx = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{x(x-2)} + \lim_{s\to 0^+} \int_s^1 \frac{dx}{x(x-2)}$

$= \lim_{t\to 0^-} \frac{1}{2} \int_{-1}^t \frac{1}{x-2} - \frac{1}{x} dx + \lim_{s\to 0^+} \frac{1}{2} \int_s^1 \frac{1}{x-2} - \frac{1}{x} dx$

$= \lim_{t\to 0^-} \frac{1}{2} \ln\left|\frac{x-2}{x}\right| \Big|_{-1}^t + \lim_{s\to 0^+} \frac{1}{2} \ln\left|\frac{x-2}{x}\right| \Big|_s^1$

$= \lim_{t\to 0^-} \frac{1}{2}\left[\ln\left|\frac{t-2}{t}\right| - \ln 3 \right] + \lim_{s\to 0^+} \frac{1}{2}\left[\ln|-1| - \ln\left|\frac{s-2}{s}\right| \right]$

$= \infty - \infty$ divergent.

(d) $\int \frac{3x^2-2}{x^2-2x-8}dx = \int 3 + \frac{6x+22}{x^2-2x-8}\,dx = \int 3 + \frac{3\,(2x-2)}{x^2-2x-8} + \frac{28}{x^2-2x-8}\,dx$

$$= \int 3 + \frac{3\,(2x-2)}{x^2-2x-8} + \frac{28}{6}\left(\frac{1}{x-4} - \frac{1}{x+2}\right) dx$$

$$= 3x + 3\ln|x^2-2x-8| + \frac{14}{3}\left(\ln|x-4| - \ln|x+2|\right) + C.$$

$$= 3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C.$$

(e) $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}dx = -\frac{\sqrt{1+x^2}}{x} - \int -\frac{1}{\cancel{x}} \frac{\cancel{x}}{\sqrt{1+x^2}}\,dx$

$\sqrt{1+x^2} = u \Rightarrow du = \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\,dx$

$\frac{dx}{x^2} = dv \Rightarrow v = -\frac{1}{x}$

$x = \tan\theta \quad dx = \sec^2\theta\, d\theta$

$\int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}} = \ln|\sec\theta + \tan\theta|$

$= \ln\left|\sqrt{1+x^2} + x\right|$

$$= -\frac{\sqrt{1+x^2}}{x} + \ln\left|\sqrt{1+x^2} + x\right| + C$$

(f) $\int x^5 e^{-x^3}dx \underset{x^3 = s}{=} \int s\, e^{-s} \cdot \frac{ds}{3} = \frac{1}{3}\left[-se^{-s} - \int -e^{-s}ds\right]$

$3x^2dx = ds$

$s = u \Rightarrow du = ds$

$e^{-s}ds = dv \Rightarrow v = -e^{-s}$

$$= \frac{1}{3}\left[-se^{-s} - e^{-s}\right] + C$$

$$= -\frac{(x^3+1)}{3}e^{-x^3} + C.$$

6. *(10pts)* Show that $f(x) = x^2 - 2x - \frac{\sin}{2}\left(\frac{\pi x}{2}\right)$ has exactly 2 roots.

$f(x)$ has at least two roots: $f(0) = f(2) = 0$

$f(x)$ can not have more than two roots! Since $f(x)$ is difference of polynomial and sine function, it is differentiable and continuous everywhere.

If f has more than two roots, we can choose 3 say $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = 0$. Using MVT on $[\alpha_1, \alpha_2]$ and $[\alpha_2, \alpha_3]$ there exists $\beta_1 \in [\alpha_1, \alpha_2]$, $\beta_2 \in [\alpha_2, \alpha_3]$ such that $f'(\beta_1) = f'(\beta_2) = 0$

Now, using MVT on $[\beta_1, \beta_2]$ (f' is also differentiable) there exists $c \in [\beta_1, \beta_2]$ such that $f''(c) = 0$

But $f''(c) = 2 + \frac{\pi^2}{8} \sin\left(\frac{\pi x}{2}\right) \geq 2 - \frac{\pi^2}{8} > 0$.

Therefore, f has exactly two roots.

Bonus. *(3pts)* Compute $\int \frac{1}{x^7 - x} dx$.

$$\int \frac{1}{x^7 - x} dx = \int \frac{x^2 \, dx}{x^9 - x^3} \underset{\substack{x^3 = u \\ 3x^2 dx = du}}{=\!=\!=} \frac{1}{3} \int \frac{du}{u^3 - u} = \frac{1}{3} \int \frac{du}{u(u-1)(u+1)}$$

$$= \frac{1}{6} \int \frac{1}{u-1} + \frac{1}{u+1} - \frac{2}{u} \, du$$

$$= \frac{1}{6} \left[\ln(u-1) + \ln(u+1) - 2\ln u \right]$$

$$= \frac{1}{6} \ln \left| \frac{x^6 - 1}{x^6} \right| + C.$$