METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM		
Code : *MAT 119*	Last Name:	
Acad.Year: *2011-2012*	Name :	Student No.:
Semester : *SUMMER*	Department:	Section:
Date : *21.7.2012*	Signature:	
Time : *13:30*	9 QUESTIONS ON 8 PAGES	
Duration : *100 minutes*	TOTAL 100 POINTS	

1	2	3	4	5	6	7	8	9		

Show your work! Please draw a $\boxed{\textbf{box}}$ around your answers!

1. Find the value of a which makes $f(x)$ continuous if $f(x) = \begin{cases} |x-2|, & \text{if } x \le 1 \\ x^2+5x+a, & \text{if } x > 1. \end{cases}$

$$\lim_{x\to 1^-} |x-2| = \lim_{x\to 1^-} -(x-2) = 1$$

$$\lim_{x\to 1^+} x^2+5x+a = 1+5+a = 6+a$$

Need $1 = 6+a$

$$\boxed{a=-5}$$

2. Find the following limits, if they exist. Show your work. **Do not use L'Hospital's rule.**

(a) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x+1$$

$$= \boxed{2}$$

(b) $\lim_{x\to 5} \dfrac{|x-5|}{x^2-5x}$

$$\lim_{x\to 5^-} \frac{|x-5|}{x^2-5x} = \lim_{x\to 5^-} \frac{-\cancel{(x-5)}}{x\cancel{(x-5)}} = -1/5$$

$$\lim_{x\to 5^+} \frac{|x-5|}{x^2-5x} = \lim_{x\to 5^+} \frac{\cancel{(x-5)}}{x\cancel{(x-5)}} = 1/5 \quad *$$

$\boxed{\lim_{x\to 5} \frac{|x-5|}{x^2-5x} \text{ DNE}}$ because left and right limits are different

(c) $\lim_{x\to -\infty} \dfrac{\sqrt{3x^2+1}}{2x-5}$

$$\lim_{x\to -\infty} \frac{\sqrt{3x^2+1}}{2x-5} = \lim_{x\to -\infty} \frac{\sqrt{3+1/x^2}}{2-5/x}\cdot\frac{\sqrt{x^2}}{x}$$

If $x<0$ $\sqrt{x^2} = -x$

$$= \frac{\sqrt{3}}{2}(-1)$$

$$= \boxed{-\sqrt{3}/2}$$

(d) $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$

$$\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\cdot\frac{\sqrt{x}+x^2}{\sqrt{x}+x^2} = \lim_{x\to 1} \frac{x-x^4}{1-x}\cdot\frac{1+\sqrt{x}}{\sqrt{x}+x^2}$$

$$= \lim_{x\to 1} \frac{x\cancel{(1-x)}(1+x+x^2)}{\cancel{(1-x)}}\cdot\frac{1+\sqrt{x}}{\sqrt{x}+x^2}$$

$$= (1\cdot 3)\cdot\frac{\cancel{1+1}}{\cancel{1+1}} = \boxed{3}$$

3. Give a formal ε-δ proof that $\lim_{x\to 3}(x^2+2x-1) = 14$.

Proof:

Let $\varepsilon > 0$ and $\delta = \min(1, \varepsilon/9)$.

If $0 < |x-3| < \delta$ then

① $|x-3| < 1$

$-1 < x-3 < 1$

$7 < x+5 < 9$

$|x+5| < 9$

② $|x-3| < \varepsilon/9$

$|x+5| \cdot |x-3| < 9|x-3| < \varepsilon$

$|x^2+2x-15| < \varepsilon$

$|(x^2+2x-1)-14| < \varepsilon$

So $|f(x)-14| < \varepsilon$.

Scratch work:

$|x^2+2x-1-14| < \varepsilon$

$|x^2+2x-15| < \varepsilon$

$|x+5||x-3| < \varepsilon$

$|x-3| < \dfrac{\varepsilon}{|x+5|}$

If $|x-3| < 1$

then

$-1 < x-3 < 1$

$2 < x < 4$

$7 < x+5 < 9$

$|x+5| < 9$

4. Calculate the following derivatives.

(a) $\dfrac{d}{dx}\left(x^2 \sin(x)\cos(x)\right)$

$= \left(\frac{d}{dx}x^2\right)\sin x \cos x + x^2\left(\frac{d}{dx}\sin x\right)\cos x + x^2 \sin x\left(\frac{d}{dx}\cos x\right)$

$= \boxed{2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x}$

$x\sin 2x + x^2\cos 2x$ is also correct.

(b) $\dfrac{d}{dx}\left((119x+3)^{2012}\right)$

$= 2012\,(119x+3)^{2011}\cdot\left(\frac{d}{dx}\,119x+3\right)$

$= \boxed{2012\,(119x+3)^{2011}\cdot 119}$

(c) $\dfrac{d}{dx}\left(x\tan(x^2-1)\right)$

$= \left(\frac{d}{dx}x\right)\tan(x^2-1) + x\left(\frac{d}{dx}\tan(x^2-1)\right)$

$= \tan(x^2-1) + x\left(\sec^2(x^2-1)\left(\frac{d}{dx}x^2-1\right)\right)$

$= \boxed{\tan(x^2-1) + x\sec^2(x^2-1)\cdot(2x)}$

(d) $\dfrac{d}{dx}\left(\dfrac{\sec(\sqrt{x})}{x^3-2}\right)$

$= \boxed{\dfrac{\left(\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2}\frac{1}{\sqrt{x}}\right)(x^3-2) - \sec\sqrt{x}\,(3x^2)}{(x^3-2)^2}}$

5. Find the equation of the tangent line to $x^2 + 3x^2y^2 + y^3 = 5$ at the point $(1, 1)$.

First find slope $= y'$:

$$x^2 + 3x^2y^2 + y^3 = 5$$

$$\downarrow \tfrac{d}{dx}$$

$$2x + 6xy^2 + 6x^2yy' + 3y^2y' = 0$$

$$(6x^2y + 3y^2)y' = -(2x + 6xy^2)$$

$$y' = -\frac{2x + 6xy^2}{6x^2y + 3y^2}$$

@ (1,1): $y'(1,1) = -\dfrac{2+6}{6+3} = -8/9$

Tangent line equation:

$$\boxed{y = -8/9(x-1) + 1}$$

slope $= -8/9$, through point $(1,1)$

6. Suppose f is a continuous function and $f'(x)$ exists everywhere. If $f(2) = 10$ and $f'(x) \geq -3$ for all x then what is the smallest possible value for $f(4)$?

By the mean value thm there is c between 2 and 4 with

$$f'(c) = \frac{f(4) - f(2)}{4-2} = \frac{f(4) - 10}{2}$$

But $-3 \leq f'(c)$, so

$$-3 \leq \frac{f(4)}{2} - 5$$

$$\boxed{4 \leq f(4)}$$

7. The volume of a cube grows at a constant rate of $2\,\frac{\text{cm}^3}{\text{min}}$.

(a) Compute the rate a side of the cube is growing at the moment the side length is $2\,\text{cm}$.

$$\text{Vol} = x^3$$

$$\frac{d}{dt}\text{Vol} = 2\ \text{cm}^3/\text{min}$$

$$x(t_0) = 2\ \text{cm}$$

$$\frac{d}{dt}$$

$$\left(\frac{d}{dt}\text{Vol}\right) = 3x^2\left(\frac{d}{dt}x\right)$$

$$\left(\frac{d}{dt}x\right) = \left(\frac{d}{dt}\text{Vol}\right)\frac{1}{3x^2}$$

at time t_0: $$\left(\frac{d}{dt}x\right) = 2\left(\text{cm}^3/\text{min}\right)\frac{1}{3\cdot 4\ \text{cm}^2}$$

$$= \boxed{1/6\ \text{cm}/\text{min}}$$

(b) Compute the rate the surface area is growing at the moment the side length is $2\,\text{cm}$.

$$SA = 6x^2$$

$$d/dt$$

$$\left(\frac{d}{dt}x\right)(t_0) = 1/6\ \text{cm}/\text{min}$$

$$x(t_0) = 2\ \text{cm}$$

$$\left(\frac{d}{dt}SA\right) = 12x\left(\frac{d}{dt}x\right)$$

at time t_0: $$\left(\frac{d}{dt}SA\right) = 12\cdot(2\ \text{cm})(1/6\ \text{cm}/\text{min})$$

$$= \boxed{4\ \text{cm}^2/\text{min}}$$

8. Consider the function $f(x) = \frac{2x+4}{x-1}$

$\boxed{(0,-4) \text{ \& } (-2,0)}$

(a) Give the domain and intercepts.

Domain: $x-1 \neq 0 \leadsto \boxed{x \neq 1 \text{ aka. } (-\infty,1) \cup (1,\infty)}$

Intercepts: $x=0 \leadsto y = 4/-1 = -4$ $\quad$ $y=0 \leadsto 0 = \frac{2x+4}{x-1} \leadsto x=-2$

(b) Find vertical, horizontal, or slant asymptotes.

Vertical: $\lim_{x\to 1^-} \frac{2x+4}{x-1} = -\infty$

$\lim_{x\to 1^+} \frac{2x+4}{x-1} = \infty$

$\boxed{\text{V. asympt. at } x=1}$

Horizontal: $\lim_{x\to\pm\infty} \frac{2x+4}{x-1} = 2$ $\quad$ $\boxed{\text{H. asympt. at } y=2}$

(c) Find the intervals of increase and decrease and local max/min.

$f = \frac{2x+4}{x-1} = 2 + 6/(x-1)$

$2x+4 \,/\, (x-1)$: $\quad 2 + 6/(x-1)$

$2x-2$

6

$f' = 0 + \frac{-6(1)}{(x-1)^2}$

$= -6/(x-1)^2$ always negative

$\boxed{\Rightarrow f \text{ always decreasing } \left(\begin{matrix}\text{No max/min} \\ \text{No increase}\end{matrix}\right)}$

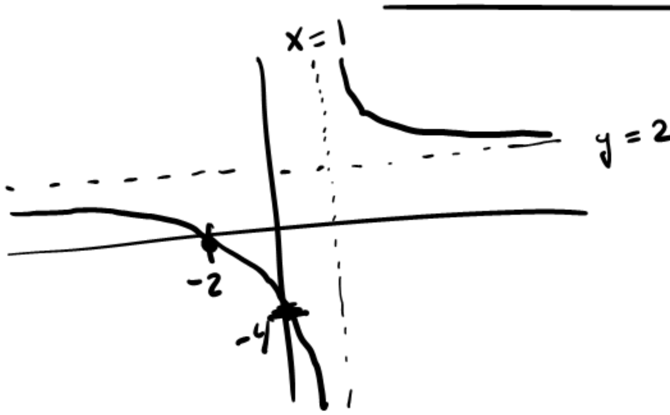
(d) Find the intervals of where f is concave up or down and inflection points.

$f' = -6/(x-1)^2$

$f'' = \frac{6 \cdot 2(x-1)\cdot 1}{(x-1)^4} = 12 \cdot \frac{(x-1)}{(x-1)^4} = 12/(x-1)^3$

f'' is never $0 \Rightarrow$ no inflection points.
$f'' < 0$ for $x<1 \Rightarrow$ concave down.
$f'' > 0$ for $x>1 \Rightarrow$ concave up.

(e) Draw the a rough graph of f below.



9. A rectangle has sides with length x and y. Find the maximum area if the sides satisfy $x = -(y^2 + 3y - 9)$.

$$\text{Area} = xy$$

$$x = -(y^2 + 3y - 9)$$

$$\text{Area} = -(y^2 + 3y - 9)y$$

$$= -y^3 - 3y^2 + 9y$$

critical pts:

$$0 = \left(\frac{d}{dy}\text{Area}\right) = -3y^2 - 6y + 9$$

$$0 = -3(y^2 + 2y - 3)$$

$$0 = -3(y+3)(y-1)$$

$$y = 1, -3$$

|| $y = -3$ does not make sense as a side length

|| $\frac{d^2}{dy^2}\text{Area} = -6y - 6 \leadsto = -2$ @ $y = 1$

So $y = 1$ is local max for Area.

$$y = 1 \Longrightarrow x = -(1 + 3 - 9) = 5$$

$$\text{Max Area} = 1 \cdot 5 = \boxed{5}$$