METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL EXAM	
Code : *MAT 119*	Last Name:
Acad.Year: *2011-2012*	Name : Solutions
	Student No.:
Semester : *SUMMER*	Department:
	Section:
Date : *10.8.2012*	Signature:
Time : *9:40*	8 QUESTIONS ON 7 PAGES
Duration : *150 minutes*	TOTAL 100 POINTS

1 (6)	2 (6)	3 (15)	4 (16)	5 (12)	6 (5)	7 (10)	8 (30)		

Show your work! Please draw a $\boxed{\text{box}}$ **around your answers!**

1. *(6 pts)* Write definite integrals expressing the area between $f(x) = x\ln(x^2+1)$ and $g(x) = 3\ln(x^2+1)$ from $x=-1$ to $x=4$.
Only write the integrals (DO NOT INTEGRATE).

Crossing points: $x\ln(x^2+1) = 3\ln(x^2+1)$

$(x-3)\ln(x^2+1) = 0 \longrightarrow \begin{cases} x-3=0 \Rightarrow x=3 \\ \ln(x^2+1)=0 \Rightarrow x=0 \end{cases}$

$$\text{Area} = \int_{-1}^{4} \left| x\ln(x^2+1) - 3\ln(x^2+1) \right| dx$$

$$= \int_{-1}^{4} |x-3| \cdot |\ln(x^2+1)|\, dx$$

$$= \boxed{\int_{-1}^{0} (3-x)(\ln(x^2+1))\,dx + \int_{0}^{3} (3-x)\ln(x^2+1)\,dx + \int_{3}^{4} (x-3)\ln(x^2+1)\,dx}$$

$$\left(= \left|\int_{-1}^{0}(x-3)\ln(x^2+1)\,dx\right| + \left|\int_{0}^{3}(x-3)\ln(x^2+1)\,dx\right| + \left|\int_{3}^{4}(x-3)\ln(x^2+1)\,dx\right| \right)$$

2. *(6 pts)* Write a definite integral expressing the arc length of $f(x) = x\ln x$ from $x=2$ to $x=3$.
Only write the integrals (DO NOT INTEGRATE).

$f'(x) = \ln x + \frac{x}{x}$

$= \ln x + 1$

$$\text{Arc Length} = \int_{x=2}^{x=3} 1\, ds$$

$$= \int_{2}^{3} \sqrt{(f')^2+1}\, dx$$

$$= \boxed{\int_{2}^{3} \sqrt{(\ln x + 1)^2 + 1}\, dx}$$

3. *(5×3 pts)* Calculate the following limits.

(a) $\lim_{x\to 0} \dfrac{\ln(x+1)-x}{x^2}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{x+1}-1}{2x} = \lim_{x\to 0} \frac{1-(x+1)}{2x(x+1)}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-1}{4x+2}$$

$$= \boxed{\frac{-1}{2}}$$

$\frac{\ln(1)-0}{0} = \frac{0}{0}$ $\quad$ $0/0$

(b) $\lim_{x\to -\infty} x\,e^{x} = \lim_{x\to-\infty} \dfrac{x}{e^{-x}} \overset{L'H}{=} \lim_{x\to-\infty} \dfrac{1}{-e^{-x}} = \boxed{0}$

$\frac{-\infty}{e^{\infty}} = \frac{\infty}{\infty}$

(c) $\lim_{x\to\infty} \sqrt{x^2+3x+3} - x$

$$= \lim_{x\to\infty} \sqrt{x^2+3x+3} - x \cdot \left(\frac{\sqrt{x^2+3x+3}+x}{\sqrt{x^2+3x+3}+x}\right)$$

$\infty - \infty$

$$= \lim_{x\to\infty} \frac{(x^2+3x+3)-x^2}{\sqrt{x^2+3x+3}+x}$$

$\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{3+3/x}{\sqrt{1+3/x+3/x^2}+1}\left(\frac{x}{x}\right) = \boxed{3/2}$$

This problem is easier to solve without L'Hospital's Rule!!

(d) $\lim_{x\to 0^+} x^{3\sin x}$

$$= \lim_{x\to0^+} e^{\ln x^{3\sin x}} = \lim_{x\to 0^+} e^{3\sin x \ln x} = \boxed{1}$$

$$\lim_{x\to0^+} 3\sin x \ln x = \lim_{x\to0^+} \frac{3\ln x}{\frac{1}{\sin x}} \overset{L'H}{=} \lim_{x\to0^+} \frac{3/x}{-\csc x \cot x}$$

$0\cdot(-\infty)$ $\quad$ $\frac{-\infty}{\infty}$

$$= \lim_{x\to0^+} -\frac{3\sin x \tan x}{x}$$

$0/0$

$$\overset{L'H}{=} \lim_{x\to0^+} -\frac{3(\cos x\tan x + \sin x \sec^2 x)}{1} = 0/1$$

$e^0 = 1$

(e) $\lim_{x\to 0^+} x^{\cos x}$

$$= \lim_{x\to0^+} e^{\ln x^{\cos x}} = \lim_{x\to0^+} e^{\cos x \ln x} = \boxed{0}$$

$$\lim_{x\to0^+} \cos x \ln x = -\infty$$

$1\cdot \ln 0 = 1\cdot(-\infty)$

"$e^{-\infty} = 0$"

4. *(4×4 pts)* Calculate the following derivatives.

(a) $\frac{d}{dx}\left(\ln\left(\sqrt{\frac{2x+1}{x-2}}\right)\right) = \frac{d}{dx}\left(\frac{1}{2}\left(\ln(2x+1) - \ln(x-2)\right)\right)$

$$= \boxed{\frac{1}{2}\left(\frac{1}{2x+1}\cdot 2 - \frac{1}{x-2}\right)}$$

(b) $\frac{d}{dx}\left(\arcsin(2^{(x^2+1)})\right) = \frac{1}{\sqrt{1-(2^{x^2+1})^2}}\cdot\frac{d}{dx}\left(2^{x^2+1}\right)$

$$= \frac{1}{\sqrt{1-(2^{2x^2+2})}}(\ln 2)\, 2^{x^2+1}\frac{d}{dx}(x^2+1)$$

$$= \boxed{\frac{1}{\sqrt{1-(2^{2x^2+2})}}(\ln 2)\, 2^{x^2+1}(2x)}$$

(c) $\frac{d}{dx}\left(x^{(x^x)}\right)$

Recall: Logarithmic Diff:

$\frac{d}{dx}\ln y = \frac{y'}{y}$

$\Downarrow$

$y' = y\left(\frac{d}{dx}\ln y\right)$

$\frac{d}{dx}x^{x^x} = x^{x^x}\left(\frac{d}{dx}\ln x^{x^x}\right) = x^{x^x}\left(\frac{d}{dx}x^x\ln x\right)$ (Log diff)

(product rule) $= x^{x^x}\left(\left(\frac{d}{dx}x^x\right)\cdot\ln x + x^x\cdot\frac{1}{x}\right)$

(Log. diff) $= x^{x^x}\left(x^x\left(\frac{d}{dx}\ln x^x\right)\ln x + x^x\cdot\frac{1}{x}\right)$

(product rule) $= x^{x^x}\left(x^x\cdot(\ln x+1)\ln x + x^x\cdot\frac{1}{x}\right)$

$$= \boxed{x^{x^x+x}\cdot\left((\ln x)^2+\ln x\right) + x^{x^x+x-1}}$$

(d) $\frac{d}{dx}\left(\int_{\sqrt{x}-1}^{x^2+2}\arccos(t^2)\,dt\right)$

$\parallel$

$$\boxed{\arccos\left((x^2+2)^2\right)\cdot \underbrace{2x}_{\frac{d}{dx}(x^2+2)} - \arccos\left((\sqrt{x}-1)^2\right)\cdot\underbrace{\frac{1}{2\sqrt{x}}}_{\frac{d}{dx}(\sqrt{x}-1)}}$$

5. *(3×4 pts)* Let R be the region enclosed by the curves $y = x^3 + x + 1$ and $y = 2x^2 + 1$. In the parts below your answer should be a definite integral. (DO NOT INTEGRATE.)

(a) Write a definite integral which computes the volume of the solid formed by rotating R around the x-axis.

Crossing Points:

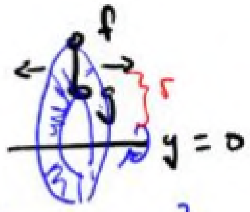
$x^3 + x + 1 = 2x^2 + 1$

$x^3 - 2x^2 + x = 0$

$x(x-1)^2 = 0$

$x = 0, 1$

between $0 \le 1$ this is > 0 so $x^3 + x + 1 > 2x^2 + 1$



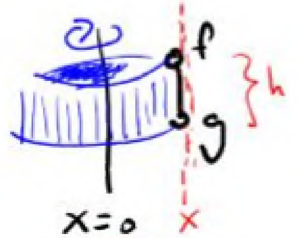
Area $= \pi r^2 = \pi(f^2 - g^2)$

$$\text{Volume} = \int_0^1 \pi \left| (x^3 + x + 1)^2 - (2x^2+1)^2 \right| dx$$

$$= \left| \int_0^1 \pi \left((x^3+x+1)^2 - (2x^2+1)^2 \right) dx \right|$$

$$= \int_0^1 \pi \left((x^3+x+1)^2 - (2x^2+1)^2 \right) dx$$

(b) Write a definite integral which computes the volume of the solid formed by rotating R around the y-axis.



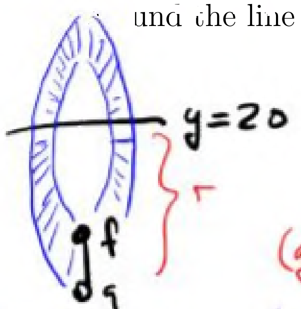
Area $= 2\pi r h = 2\pi x (f - g)$

$$\text{Volume} = \int_0^1 2\pi x \left| (x^3+x+1) - (2x^2+1) \right| dx$$

$$= \left| \int_0^1 2\pi x \left((x^3+x+1) - (2x^2+1) \right) dx \right|$$

$$= \int_0^1 2\pi x \left((x^3+x+1) - (2x^2+1) \right) dx$$

(c) Write a definite integral which computes the volume of the solid formed by rotating R around the line $y = 20$.



$(g-20)^2 \quad (f-20)^2$

$\quad\;\; \| \qquad\quad\; \|$

Area $= \pi r^2 = \pi\left((20-g)^2 - (20-f)^2 \right)$

$$\text{Volume} = \int_0^1 \pi \left| (x^3+x+1-20)^2 - (2x^2+1-20)^2 \right| dx$$

$$= \left| \int_0^1 \pi \left((x^3+x-19)^2 - (2x^2-19)^2 \right) dx \right|$$

$$= \int_0^1 \pi \left((2x^2-19)^2 - (x^3+x-19)^2 \right) dx$$

6. *(5 pts)* Find the absolute maximum and minimum values of

$$f(x) = \frac{3x^2}{2} - 7x - \frac{4}{x}$$

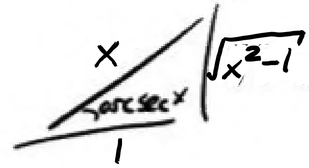
on the interval $[-1, 1]$.

$$\lim_{x \to 0^+} \frac{3x^2}{2} - 7x - \frac{4}{x} = -\infty \quad \text{so } \underline{\underline{\text{no}}} \text{ abs min!}$$

$$\lim_{x \to 0^-} \frac{3x^2}{2} - 7x - \frac{4}{x} = +\infty \quad \text{so } \underline{\underline{\text{no}}} \text{ abs max!}$$

7. *(2×5 pts)* In this problem you will compute the integral of arcsec(x).

(a) Let $f(x) = \sec(x)$ so that $f^{-1}(x) = \text{arcsec}(x)$. Use this to derive the formula for $\frac{d}{dx}\left(\text{arcsec}(x)\right)$.



$$f(x) = \sec x$$

$$f'(x) = \sec x \tan x$$

$$\frac{d}{dx} f^{-1} = \frac{1}{f'(f^{-1})} = \frac{1}{\sec(\text{arcsec}\,x)\tan(\text{arcsec}\,x)}$$

$$= \boxed{\frac{1}{x\sqrt{x^2-1}}}$$

(b) Use your answer from (a) to compute $\int \text{arcsec}(x)\, dx$.

$$\int \text{arcsec}\,x\, dx$$

$u = \text{arcsec}\,x \quad du = \frac{1}{x\sqrt{x^2-1}}$

$dv = dx \quad v = x$

$$= x\,\text{arcsec}\,x - \int \frac{\cancel{x}}{\cancel{x}\sqrt{x^2-1}}\, dx$$

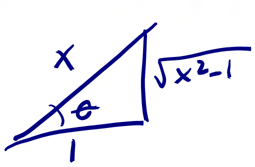
$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \tan\theta$

$$= x\,\text{arcsec}\,x - \int \frac{1}{\cancel{\tan\theta}} \sec\theta \cancel{\tan\theta}\, d\theta$$

$$= x\,\text{arcsec}\,x - \ln|\sec\theta + \tan\theta| + C$$



$$= \boxed{x\,\text{arcsec}\,x - \ln\left|x + \sqrt{x^2-1}\right| + C}$$

8. *(6×5 pts)* Compute the following integrals.

(a) $\int \sec^3 x \tan^3 x \, dx$

$$\left.\begin{aligned} u &= \sec x \\ du &= \sec x \tan x \, dx \\ 1-u^2 &= \tan^2 x \end{aligned}\right\} = \int u^2 (1-u^2)\, du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{3}\sec^3 x - \frac{1}{5}\sec^5 x + C}$$

(b) $\int_1^2 \frac{\sqrt{x+1}}{x}\, dx$

Long division

$$\left.\begin{aligned} u^2 &= x+1 \\ u^2-1 &= x \\ 2u\,du &= dx \end{aligned}\right\} = \int_{\sqrt{2}}^{\sqrt{3}} \frac{u}{u^2-1} \cdot 2u \, du = \int_{\sqrt{2}}^{\sqrt{3}} 2 + \frac{2}{u^2-1}\, du$$

$A(u-1) + B(u+1) = 2$

$u=1$: $B \cdot 2 = 2 \leadsto B = 1$

$u=-1$: $A(-2) = 2 \leadsto A = -1$

$$= \int_{\sqrt{2}}^{\sqrt{3}} 2 + \frac{A}{u+1} + \frac{B}{u-1}\, du$$

$$= 2u - \ln|u+1| + \ln|u-1| \Big|_{\sqrt{2}}^{\sqrt{3}}$$

$$= \boxed{\begin{aligned} &2\sqrt{3} - \ln|\sqrt{3}+1| + \ln|\sqrt{3}-1| \\ &\quad - 2\sqrt{2} + \ln|\sqrt{2}+1| - \ln|\sqrt{2}-1| \end{aligned}}$$

(c) $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$

$$\left.\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}\right\} = \int_0^1 e^u \, du = e^u \Big|_0^1$$

$$= \boxed{e - 1}$$

(d) $\int (x^2+2x+5)^{-1.5}\,dx$ $= \int \frac{1}{(x^2+2x+5)^{3/2}}\,dx = \int \frac{dx}{((x+1)^2+4)^{3/2}}$

$\frac{x+1}{2} = \tan\theta$

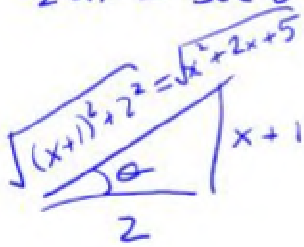
$\sqrt{(x+1)^2+4} = \sqrt{4\sec^2\theta}$

$= 2\sec\theta$

$\frac{1}{2}dx = \sec^2\theta\, d\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{(2\sec\theta)^3}$$

$$= \int \frac{1}{4}\frac{1}{\sec\theta}\,d\theta = \int \frac{1}{4}\cos\theta\, d\theta$$



$$= \frac{1}{4}\sin\theta + C$$

$$= \boxed{\frac{1}{4}\frac{x+1}{\sqrt{x^2+2x+5}} + C}$$

(e) $\int \frac{3x^3-x^2-4x-6}{x^4+2x^3+2x^2}\,dx$

$$= \int \frac{3x^3-x^2-4x-6}{x^2(x^2+2x+2)}\,dx$$

Type III Partial Fractions

$$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{C(2x+2)}{x^2+2x+2} + \frac{D}{((x+1)^2+1)}\,dx$$

$A=1$, $B=-3$, $C=1$, $D=-2$

(complete the square)

$Ax(x^2+2x+2) + B(x^2+2x+2) + C(2x+2)x^2 + Dx^2 = 3x^3-x^2-4x-6$

$\underline{x=0}$: $B(2) = -6 \rightsquigarrow B=-3$

$\underline{x\text{ coeff}}$: $2A+2B=-4 \rightsquigarrow A=1$

$\underline{x^3\text{coeff}}$: $A+2C=3 \rightsquigarrow C=1$

$\underline{x^2\text{coeff}}$: $2A+B+2C+D=-1 \rightsquigarrow D=-2$

$$= \boxed{\ln|x| + \frac{3}{x} + \ln|x^2+2x+2| - 2\arctan(x+1) + C}$$

(f) $\int_0^{\pi/4} \frac{\cos x}{x}\,dx$

Improper Integral! ($\frac{\cos x}{x}$ not defined @ 0)

Between 0 and $\pi/4$, $\cos x > \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} > 0$

So $\frac{\cos x}{x} > \frac{1/\sqrt{2}}{x} > 0$

Comparison Thm

$$\int_0^{\pi/4} \frac{\cos x}{x}\,dx > \int_0^{\pi/4} \frac{1}{\sqrt{2}}\cdot\frac{1}{x}\,dx = \infty$$

Thus $\int_0^{\pi/4} \frac{\cos x}{x}\,dx = \infty$ (Divergent)