Northern Cyprus Campus

<table>
<tr><td colspan="2">Introduction to Differential Equations
Midterm II</td></tr>
<tr><td>Code : Math 219
Acad.Year: 2013-2014
Semester : Fall
Date : 03.12.2013
Time : 17:40
Duration : 120 minutes</td><td>Last Name: KEY
Name:
Department:
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Student No:
Section:

5 QUESTIONS ON 5 PAGES
TOTAL 100 POINTS</td></tr>
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1 (15)	2 (25)	3 (15)	4 (20)	5 (25)	

Show your work! No calculators! Please draw a box around your answers! Please do not write on your desk!

1.(5+10=15 pts) This problem has two unrelated parts.

(a) Check that $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 4 & 1.5 & 0 & 0 \\ 1 & 2.5 & 2 & 0 \\ -1 & 0 & 5 & 1.5 \\ 0.5 & 1 & 0 & 4 \end{bmatrix}$$

What is the corresponding eigenvalue? (Do not compute the characteristic polynomial.)

Put $\vec{x} = [1\ 1\ 1\ 1]^T$. Then

$$A\vec{x} = 5.5\,\vec{x} \Rightarrow \lambda = 5.5 \in \sigma(A).$$

(b) Compute e^A as a single real valued matrix if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} i\pi & 0 \\ 0 & \pi/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^{-1} = PJP^{-1}$$

$$e^A = Pe^JP^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} e^{i\pi} & 0 \\ 0 & e^{\pi/2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \frac{1}{3}$$

$$= \begin{bmatrix} e^{i\pi} & 2e^{\pi/2} \\ 0 & 3e^{\pi/2} \end{bmatrix} \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} e^{i\pi} & 2/3\, e^{\pi/2} - 2/3\, e^{i\pi} \\ 0 & e^{\pi/2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2/3\,(e^{\pi/2}+1) \\ 0 & e^{\pi/2} \end{bmatrix}$$

2.(12+5+8=25 pts) Consider the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(a) Find a fundamental matrix $\Psi(t)$ for the homogeneous equation.

$$\Delta(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) \Rightarrow \sigma(A) = \{1^{①}, 3^{①}\}.$$

For $\lambda = 1$ we have $A - 1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \vec{f}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 3$ we have $A - 3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \vec{f}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and

$$\Psi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \text{ with } W(t) = 2e^{4t}.$$

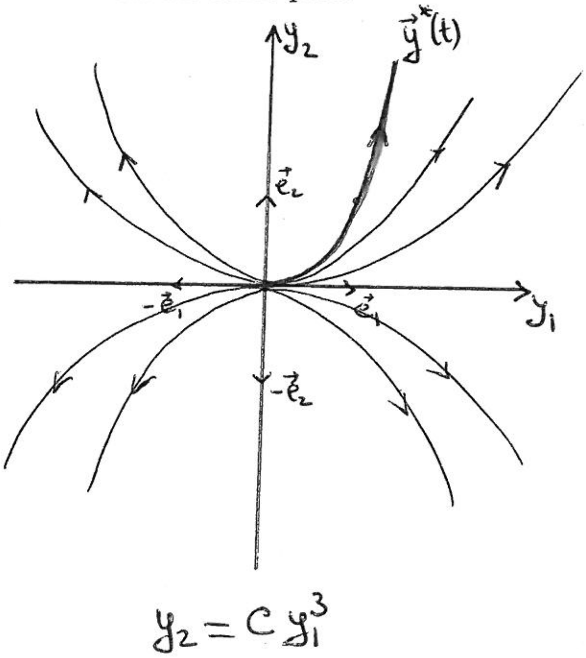
(b) Find the solution satisfying the given initial condition.

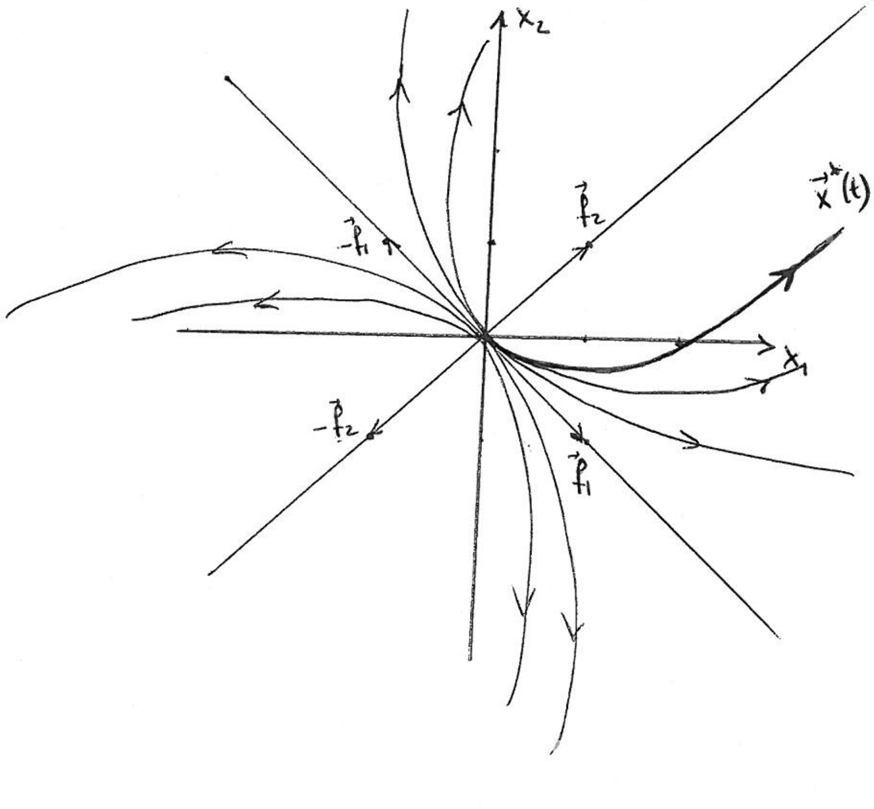
$$\Phi(t) = \Psi(t)P^{-1} = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{2} = \begin{bmatrix} e^{3t}+e^t & e^{3t}-e^t \\ e^{3t}-e^t & e^{3t}+e^t \end{bmatrix}\frac{1}{2}$$

Then $\vec{x}^*(t) = \Phi(t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{3t}+e^t & e^{3t}-e^t \\ e^{3t}-e^t & e^{3t}+e^t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{3t}+e^t \\ e^{3t}-e^t \end{bmatrix}$

Note also that $\vec{y}^*(t) = P^{-1}\vec{x}^*(t) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{3t}+e^t \\ e^{3t}-e^t \end{bmatrix} = \begin{bmatrix} e^t \\ e^{3t} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1^3 \end{bmatrix}$, $y_1 > 0$.

(c) Sketch the phase portrait of the system **AND** show the solution of the initial value problem on the same plot.





3.(5+5+5=15 pts) This problem has three unrelated parts.

(a) Suppose that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two solutions of a linear (non-homogenous) system of differential equations. Show that $\frac{1}{3}\mathbf{x}(t)+\frac{2}{3}\mathbf{y}(t)$ is also a solution.

$$\left(\frac{1}{3}\vec{x}(t)+\frac{2}{3}\vec{y}(t)\right)' = \frac{1}{3}A(t)\vec{x}(t) + \frac{1}{3}\vec{b}(t) + \frac{2}{3}A(t)\vec{y}(t) + \frac{2}{3}\vec{b}(t)$$

$$= A(t)\left(\frac{1}{3}\vec{x}(t) + \frac{2}{3}\vec{y}(t)\right) + \vec{b}(t)$$

(b) Consider a linear, homogenous system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ where the entries of $A(t)$ are continuous functions on an interval $I \subset \mathbb{R}$. Let $\mathbf{x}(t)$ be a solution of the system and a point $t_0 \in I$. Show that if $\mathbf{x}(t_0) = \vec{0}$ then $\mathbf{x}(t) = \vec{0}$ for all $t \in I$.

Consider IVP $\begin{cases} \vec{x}'(t) = A(t)\vec{x}(t) \\ \vec{x}(t_0) = \vec{0} \end{cases}$, which has only trivial solution thanks to the Existence-Uniqueness Theorem.

Whence $\vec{x}(t) = \vec{0}$ for all $t \in I$.

(c) Assume that two solutions of the system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ where $t > 0$ are

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} t \\ t \end{bmatrix}, \quad \text{and} \mathbf{x}^{(2)}(t) = \begin{bmatrix} t^{-1} \\ 3t^{-1} \end{bmatrix}$$

Find the trace of $A(t)$ ($tr(A(t)) = (A(t))_{11} + (A(t))_{22}$).

Based on Abel's formula, we have

$$W(t) = \begin{vmatrix} t & 1/t \\ t & 3/t \end{vmatrix} = 3-1=2 = C e^{\int tr(A(t))dt},$$

therefore $tr(A(t)) = 0$.

4.(20 pts) Find the general solution to the homogeneous system of differential equations below.

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

We have $\sigma(A) = \{2^{③}\}$, and $A - 2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

In particular, $V_{2,1} = \ker(A-2) = \{y = z = 0\}$ and $m(2) = 1 < 3 = alg(2)$.

But $(A-2)^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $V_{2,2} = \ker((A-2)^2) = \{z=0\}$.

Hence $\underset{\{y=z=0\}}{\underset{\parallel}{V_{2,1}}} \lneq \underset{\{z=0\}}{\underset{\parallel}{V_{2,2}}} \lneq \underset{\mathbb{C}^3}{\underset{\parallel}{V_{2,3}}}$. Choose

$\vec{f}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{f}_2 = (A-2)\vec{f}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{f}_3 = (A-2)\vec{f}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $J = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, and

$$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ \frac{t^2}{2}e^{2t} & te^{2t} & e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3t^2}{2} - t\right)e^{2t} & (3t-1)e^{2t} & 3e^{2t} \\ 3te^{2t} & 3e^{2t} & 0 \\ e^{2t} & 0 & 0 \end{bmatrix}.$$

The general solution: $\vec{x}(t) = \Psi(t)\vec{c}$.

5. (13+12=25 pts) Consider the non-homogeneous system

$$x' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \cos t \end{bmatrix} = A(t)\,\vec{x}(t) + \vec{b}(t)$$

(a) Find a fundamental matrix $\Psi(t)$ satisfying the associated homogeneous equation.

$$\Delta(\lambda) = |A - \lambda| = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1, \quad \sigma(A) = \{i, -i\}.$$

For $\lambda = i$ we have $A - i = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \sim \begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix}$,

$\vec{f} = \begin{bmatrix} 1 \\ 2/5 - i/5 \end{bmatrix} \in V_i$, and $\vec{x}(t) = \vec{f} e^{it}$ is a complex solution.

But $\vec{x}(t) = \left(\begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1/5 \end{bmatrix} \right) (\cos(t) + i\sin(t)) =$

$$= \begin{bmatrix} \cos(t) \\ 2/5 \cos(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \sin(t) \end{bmatrix} + i \left(\begin{bmatrix} 0 \\ -1/5 \cos(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ 2/5 \sin(t) \end{bmatrix} \right).$$

Hence $$\Psi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) & -\frac{1}{5}\cos(t) + \frac{2}{5}\sin(t) \end{bmatrix},$$

and $W(t) = -\frac{1}{5}\cos^2(t) + \frac{2}{5}\cos(t)\sin(t) - \frac{2}{5}\cos(t)\sin(t) - \frac{1}{5}\sin^2(t) =$
$= -1/5$.

(b) Find the general solution of the system. We have to solve the system $\Psi(t)\,\vec{c}'(t) = \vec{b}(t)$. Note that

$$c_1' = -5 \begin{vmatrix} 0 & \sin(t) \\ \cos(t) & -\frac{1}{5}\cos(t) + \frac{2}{5}\sin(t) \end{vmatrix} = 5\sin(t)\cos(t)$$

$\Rightarrow c_1 = \frac{5}{2}\sin^2(t)$.

$$c_2' = -5 \begin{vmatrix} \cos(t) & 0 \\ \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) & \cos(t) \end{vmatrix} = -5\cos^2(t)$$

$\Rightarrow c_2 = -\frac{5}{2}t - \frac{5}{4}\sin(2t)$.

Thus $\vec{y}(t) = \Psi(t) \begin{bmatrix} 5/2 \sin^2(t) \\ -5/2\, t - 5/4 \sin(2t) \end{bmatrix}$ is a special solution, and $\vec{x}(t) = \Psi(t)\,\vec{c} + \vec{y}(t)$ is the general solution.