# M E T U
## Northern Cyprus Campus

| Math 219 | Differential Equations | Midterm Exam II | 30.07.2012 |
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| Last Name<br>Name :<br>Student No KEY | Dept./Sec.:<br>Time : 17: 40<br>Duration : 90 *minutes* | Signature | |
| 4 QUESTIONS ON 4 PAGES | | | TOTAL 100 POINTS |

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**Q1 (25 = 10 + 15 p.)** Consider the following higher order linear differential equation with constant coefficients $y^{(4)} - 2y^{(3)} + 5y'' - 8y' + 4y = 0$.

(*a*) Convert it to a first order linear system $\mathbf{x}' = A\mathbf{x}$ and find the Wronskian $W(t)$ of any four solutions to the given linear differential equation.

Put $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4 = y^{(3)}$. Then $\vec{x}'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 8 & -5 & 2 \end{bmatrix}.$$ Using Abel's theorem, we derive that

$$w(t) = c\, e^{\int tr(A)dt} = c\, e^{2t} \quad (tr(A) = 2)$$

(*b*) Find the general solution to the given linear differential equation.

$$\Delta(t) = t^4 - 2t^3 + 5t^2 - 8t + 4 = (t^2+4)(t-1)^2 \Rightarrow$$
$$\Rightarrow \sigma(A) = \{1^{(2)}, 2i, -2i\}.$$

Therefore

$$y = c_1 e^t + c_2 t e^t + c_3 \cos(2t) + c_4 \sin(2t)$$

is the general solution

**Q2 (25 p.)** Based on the Variations of Parameters Method, find the general solution to the nonhomogeneous linear system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$ with $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ and $\mathbf{b}(t) = \begin{bmatrix} 2\csc(t) \\ 3\sec(t) \end{bmatrix}$, $\pi/2 < t < \pi$.

Note that $\Delta(t) = \det\begin{bmatrix} 2-t & -5 \\ 1 & -2-t \end{bmatrix} = t^2 - 4 + 5 = t^2 + 1$

$\Rightarrow \sigma(A) = \{i, -i\}$.

$A - iI = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \Rightarrow V_i = \{(2-i)x = 5y\} \Rightarrow \vec{f}^{(1)} = \begin{bmatrix} 1 \\ 2/5 - i/5 \end{bmatrix}$

The complex solution: $\vec{x}(t) = \begin{bmatrix} 1 \\ 2/5 - i/5 \end{bmatrix} e^{it} = \left(\begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1/5 \end{bmatrix}\right)$

$(\cos(t) + i\sin(t)) = \begin{bmatrix} \cos(t) \\ 2/5\cos(t) + 1/5\sin(t) \end{bmatrix} + i\begin{bmatrix} \sin(t) \\ -1/5\cos(t) + 2/5\sin(t) \end{bmatrix}$

Thus $\Psi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ 2/5\cos(t) + 1/5\sin(t) & -1/5\cos(t) + 2/5\sin(t) \end{bmatrix}$

Put $\vec{\Upsilon}(t) = \Psi(t)\vec{c}(t)$ with $\Psi(t)\vec{c}\,'(t) = \vec{b}(t)$. Since $W(t) = \det\Psi(t) = -1/5$, we obtain that

$c_1'(t) = -5\begin{vmatrix} 2\csc(t) & \sin(t) \\ 3\sec(t) & -1/5\cos(t) + 2/5\sin(t) \end{vmatrix} = -4 + 15\tan(t) + 2\cot(t),$

$c_2'(t) = -5\begin{vmatrix} \cos(t) & 2\csc(t) \\ 2/5\cos(t) + 1/5\sin(t) & 3\sec(t) \end{vmatrix} = -13 + 4\cot(t)$

Therefore $c_1(t) = -4t - 15\ln|\cos(t)| + 2\ln|\sin(t)|$ and

$c_2(t) = -13t + 4\ln|\sin(t)|$

Hence $\vec{x}(t) = \Psi(t)\vec{c} + \vec{\Upsilon}(t)$ with $\vec{\Upsilon}(t) = \Psi(t)\vec{c}(t)$, is the general solution.

**Q3 (25 p.)** Using the fundamental matrix $\Phi(t)$, find the solution to the given IVP: $\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t) \\ \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$ with the matrix $A = \begin{bmatrix} -1 & -2 \\ 2 & -5 \end{bmatrix}$. Sketch the trajectory (the phase portrait) of the solution in $x_1x_2$-plane and (**bonus 2 p.**) compute $e^A$.

$$\Delta(t) = \det \begin{bmatrix} -1-t & -2 \\ 2 & -5-t \end{bmatrix} = (t+1)(t+5) + 4 = t^2 + 6t + 9$$

$$= (t+3)^2 \Rightarrow \sigma(A) = \{-3^{(2)}\}$$

$$A + 3I = \begin{bmatrix} -1+3 & -2 \\ 2 & -5+3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \Rightarrow V_{-3} = \{x = y\}$$

So, $\dim(V_{-3}) = 1 < 2 = \text{alg mul}(-3)$. Put $\vec{f}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

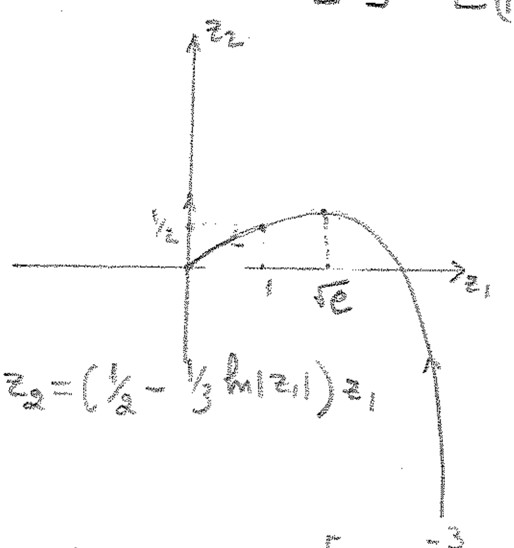
$\vec{f}^{(1)} \notin V_{-3} = \ker(A+3I)$, and $\vec{f}^{(2)} = (A+3I)\vec{f}^{(1)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

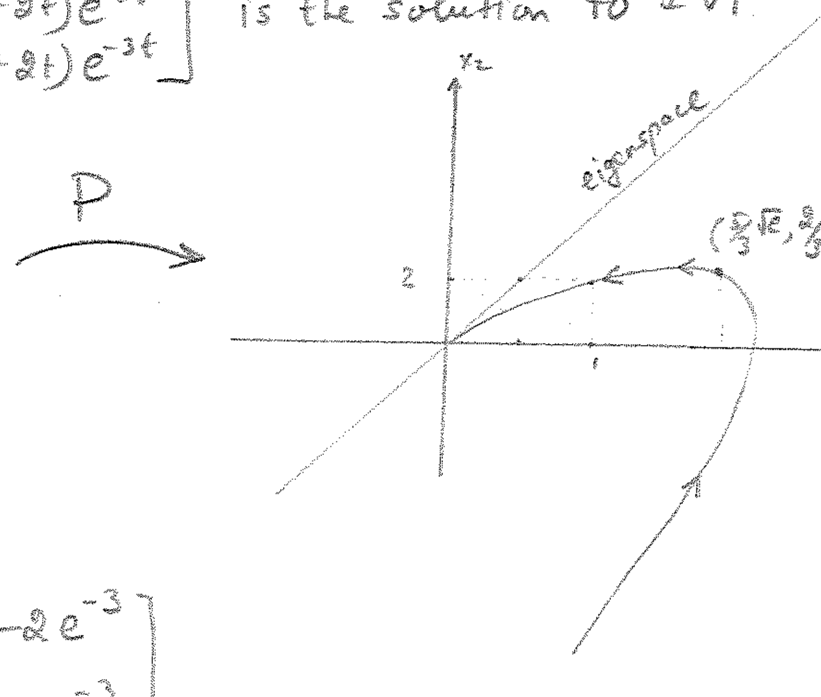
Then $P = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$, $J = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}$ and

$$\Psi(t) = P e^{Jt} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ te^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{-3t} & 2e^{-3t} \\ 2te^{-3t} & 2e^{-3t} \end{bmatrix}.$$

Moreover, $\Phi(t) = \Psi(t)P^{-1} = \begin{bmatrix} (1+2t)e^{-3t} & -2te^{-3t} \\ 2te^{-3t} & (1-2t)e^{-3t} \end{bmatrix}$ and

$\vec{x}(t) = \Phi(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (2+2t)e^{-3t} \\ (1+2t)e^{-3t} \end{bmatrix}$ is the solution to IVP




$$e^A = \Phi(1) = \begin{bmatrix} 3e^{-3} & -2e^{-3} \\ 2e^{-3} & -e^{-3} \end{bmatrix}$$

**Q4 (25 p.)** Find the fundamental matrix $\Psi(t)$ of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with the matrix $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{bmatrix}$.

$$\Delta(t) = \det \begin{bmatrix} 2-t & 2 & -1 \\ -1 & -1-t & 1 \\ -1 & -2 & 2-t \end{bmatrix} = -(t-2)^2(t+1) - 2 - 2 - 3t + 9$$

$$= -(t^3 - 3t^2 + 3t - 1) = -(t-1)^3 \Rightarrow \sigma(A) = \{1^{③}\}$$

$$\lambda = 1 \Rightarrow A - I = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix} \Rightarrow V_1 = \{x + 2y = z\} \text{ a plane}$$

$$\Rightarrow \dim(V_1) = 2 < \text{alg. mul}(1) = 3.$$

Put $\vec{f}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin V_1$, $\vec{f}^{(2)} = (A - I)\vec{f}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{f}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Therefore $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally,

$$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} =$$

$$= \begin{bmatrix} (1+t)e^t & e^t & e^t \\ -te^t & -e^t & 0 \\ -te^t & -e^t & e^t \end{bmatrix}$$