METU - NCC

	Differential Equations Midterm II	

Code : *Math 219* Acad.Year: *2011-2012* Semester : *Spring* Date : *03.5.2012* Time : *17:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature: Student No.: Section: 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (15)	2 (15)	3 (10)	4 (15)	5 (20)	6 (13)	7 (12)	

1. (10+5 pts) Consider the following homogeneous differential equation

$$y^{(5)} - y'' = 0$$

a. Find the general solution to the homogeneous equation.

The characteristic equation : $r^5 - r^2 = 0$ or $r^2(r^3-1)=0$

$\Rightarrow r = 0^{②}, 1^{①}, e^{i\,2\pi/3}, e^{i\,4\pi/3}$. But $e^{i\,2\pi/3}$ and $e^{i\,4\pi/3}$ are complex conjugate pairs, and

$e^{i\,2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3) = \underbrace{-1/2}_{\lambda} + i\,\sqrt{3}/2 = \mu$

Therefore

$$y(t) = c_1 + c_2 t + c_3 e^{t} + c_4 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_5 e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

is the general solution to the homg. equation

b. Write down the form of the particular solution $Y_p(t)$ used in the method of undetermined coefficients to solve the following non-homogeneous differential equation.
(DO NOT SOLVE)

$$y^{(5)} - y'' = 3 - te^{t}$$

Out of the duplications appeared twice, we have

$$Y_p(t) = t(At) + t(Bt+C)e^{t}$$

2. (15pts) Consider the differential equation,

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \ln(t), \qquad t > 0$$

Find the general solution if $y_1(t) = t^2$ and $y_2(t) = t^2\ln(t)$ are solutions to the corresponding homogeneous differential equation.

Put $y(t) = u_1(t)\,t^2 + u_2(t)\,t^2\ln(t)$. We have the following 2×2-system

$$\begin{cases} u_1' t^2 + u_2' t^2 \ln(t) = 0 \\ u_1' 2t + u_2'(2t\ln(t) + t) = \ln(t) \end{cases}$$

or

$$\begin{bmatrix} 1 & \ln(t) \\ 1 & \ln(t) + \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln(t)/2t \end{bmatrix}$$

Since $\det = \frac{1}{2}$, we have

$$u_1' = 2\begin{vmatrix} 0 & \ln(t) \\ \ln(t)/2t & \ln(t)+1 \end{vmatrix} = -\frac{\ln^2(t)}{t} \Rightarrow u_1 = -\int \frac{\ln^2(t)}{t}\,dt$$

$$= -\frac{\ln^3(t)}{3} + C_1, \text{ and}$$

$$u_2' = 2\begin{vmatrix} 1 & 0 \\ 1 & \ln(t)/2t \end{vmatrix} = \frac{\ln(t)}{t} \Rightarrow u_2 = \int \frac{\ln(t)}{t}\,dt$$

$= \frac{\ln^2(t)}{2} + C_2$. Hence $Y_p(t) = -\frac{1}{3}t^2\ln^3(t) + \frac{1}{2}t^2\ln^3(t)$

$= \frac{1}{6}t^2\ln^3(t)$. In particular,

$$y(t) = C_1 t^2 + C_2 t^2\ln(t) + \frac{1}{6}t^2\ln^3(t)$$

is the general solution to the diff. equation.

3. (5+5pts) This problem is about calculating the Laplace transform of $f(t) = t^2$.

a. Use the definition of the Laplace transform to show that $\mathfrak{L}\{t^2\} = \dfrac{2}{s^3}$

(*Hint:* You may use the gamma function $\Gamma(p+1) = \int_0^\infty e^{-t} t^p dt$.)

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 dt = \lim_{A\to\infty} \int_0^A e^{-st} t^2 dt = \left| \begin{array}{l} u = st \\ du = s dt \\ t = 0 \Rightarrow u = 0 \\ t = A \Rightarrow u = sA \end{array} \right|$$

$$= \lim_{A\to\infty} \int_0^{sA} e^{-u} \frac{u^2}{s^2} \frac{du}{s} = \frac{1}{s^3} \int_0^\infty e^{-u} u^2 du =$$

$$= \frac{\Gamma(3)}{s^3}. \quad \text{But } \Gamma(3) = \Gamma(2+1) = 2! \text{ Thereby}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}.$$

b. Compute again $\mathfrak{L}\{t^2\}$, but based on the convolution technique. Namely, use the definition of convolution to show $2(1\star 1\star 1) = t^2$, and show $\mathfrak{L}\{1\star 1\star 1\} = \dfrac{1}{s^3}$ by using the convolution theorem.

We have $1 * 1 = \int_0^t 1(t-\tau)\, 1(\tau)\, d\tau = \int_0^t d\tau = t,$

$t * 1 = 1 * t = \int_0^t 1(t-\tau)\, t(\tau)\, d\tau = \int_0^t \tau\, d\tau = t^2/2$, and

$1 * 1 * 1 = t^2/2$. It follows that

$$\mathcal{L}\{t^2\} = \mathcal{L}\{2 \cdot 1 * 1 * 1\} = 2\, \mathcal{L}\{1 * 1 * 1\} =$$

$$= 2\, \mathcal{L}\{1\}\, \mathcal{L}\{1\}\, \mathcal{L}\{1\} = 2\, \frac{1}{s}\, \frac{1}{s}\, \frac{1}{s} =$$

$$= \frac{2}{s^3}$$

4. (5+10pts) This problem has two unrelated parts.

a. Find the Laplace transform of $f(t) = u_2(t)e^{5t}t$ $= u_2(t)e^{5t}(t-2) +$

$+ 2u_2(t)e^{5t} = u_2(t)\, e^{5(t-2)}(t-2)\, e^{10} + 2e^{10}u_2(t)\, e^{5(t-2)}$

Therefore, $\mathcal{L}\{f(t)\} = e^{10} e^{-2s} \mathcal{L}\{e^{5t}t\} +$

$+ 2e^{10} e^{-2s} \mathcal{L}\{e^{5t}\} = e^{10} \dfrac{e^{-2s}}{(s-5)^2} + 2e^{10} \dfrac{e^{-2s}}{s-5}$.

b. Find the inverse Laplace transform of $F(s) = \dfrac{e^{-2s} + e^{-7s}}{(s-1)(s^2+4s+8)}$

Put $G(s) = \dfrac{1}{(s-1)(s^2+4s+8)}$. Then

$f(t) = u_2(t)\, g(t-2) + u_7(t)\, g(t-7)$. But

$G(s) = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+4s+8}$ with $\begin{cases} A+B=0 \\ 4A-B+C=0 \\ 8A-C=1 \end{cases}$

$\Rightarrow A = 1/13,\ B = -1/13,\ C = -5/13$. Therefore

$g(t) = \dfrac{e^t}{13} - \dfrac{1}{13}\mathcal{L}^{-1}\left\{\dfrac{s+5}{(s+2)^2+4}\right\} = \dfrac{e^t}{13} - \dfrac{1}{13}\mathcal{L}^{-1}\left\{\dfrac{s+2+3}{(s+2)^2+4}\right\}$

$= \dfrac{e^t}{13} - \dfrac{1}{13}\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+4}\right\} - \dfrac{3}{13\cdot 2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s+2)^2+4}\right\}$

$= \dfrac{e^t}{13} - \dfrac{1}{13} e^{-2t}\cos(2t) - \dfrac{3}{26} e^{-2t}\sin(2t)$

5. (20pts) Consider the initial value problem

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t), \qquad y(0) = 0, \qquad y'(0) = \frac{1}{2}$$

Find the solution to IVP by using the Laplace transform.

Put $Y(s) = \mathcal{L}\{y(t)\}$. Then $\mathcal{L}\{y'\} = sY(s)$ and $\mathcal{L}\{y''\} = s^2 Y(s) - 1/2 \Rightarrow (s^2+3s+2)Y(s) = \frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}$.

Thus

$$Y(s) = \frac{1}{2(s+1)(s+2)} + \frac{e^{-5s}}{(s+1)(s+2)} + \frac{e^{-10s}}{s(s+1)(s+2)}$$

Note that 1) $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$;

2) $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$ with

$$\begin{cases} A+B+C=0 \\ 3A+2B+C=0 \\ 2A=1 \end{cases} \Rightarrow A = \tfrac{1}{2},\ B=-1,\ C=\tfrac{1}{2}.$$

Therefore

$$Y(s) = \frac{1}{2(s+1)} - \frac{1}{2(s+2)} + \frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s+2} + \frac{e^{-10s}}{2(s+2)} + \frac{e^{-10s}}{2s} - \frac{e^{-10s}}{s+1} \Rightarrow$$

$$y(t) = \frac{e^{-t}}{2} - \frac{e^{-2t}}{2} + u_5(t)e^{-(t-5)} - u_5(t)e^{-2(t-5)} + \frac{1}{2}u_{10}(t)e^{-2(t-10)} + \frac{1}{2}u_{10}(t) - u_{10}(t)e^{-(t-10)}$$

6. (7+6pts) Let $f(t) = \begin{cases} t^2 & 0 \leqslant t < 2 \\ 1 & 2 \leqslant t \end{cases}$ be a piecewise continuous function.

a. Write down $f(t)$ as a combination of step functions $u_c(t)$.

$$f(t) = t^2 + u_2(t)\left(1 - t^2\right) = t^2 + u_2(t) - u_2(t)\left(t-2+2\right)^2 =$$
$$= t^2 + u_2(t) - u_2(t)(t-2)^2 - 4u_2(t)(t-2) - 4u_2(t)$$
$$= t^2 - 3u_2(t) - 4u_2(t)(t-2) - u_2(t)(t-2)^2$$

b. Calculate the Laplace transform $\mathfrak{L}\{f(t)\}$ by using your expression found in part (a).

$$F(s) = \frac{2}{s^3} - \frac{3e^{-2s}}{s} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2}$$

7. (5+3+4pts) This problem has three unrelated parts.

a. By using the definition of convolution, show that $f \star g = g \star f$.

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \left| \begin{array}{l} u = t - \tau \\ du = -d\tau \\ \tau = 0 \Rightarrow u = t \\ \tau = t \Rightarrow u = 0 \end{array} \right| = -\int_t^0 f(u)g(t-u)\,du$$
$$= \int_0^t g(t-u)f(u)\,du = (g * f)(t)$$

b. Find a function $f(t)$ such that $1 \star f = f$.

Put $f(t) = 0$. Then $1 * 0 = 0$.

c. Find a function $f(t)$ such that $1 \star f \neq f$

Put $f(t) = 1$. Then

$1 * 1 = t \neq 1$.