METU - NCC

	Differential Equations Midterm II	

Code : *Math 219* Acad.Year: *2011-2012* Semester : *Fall* Date : *3.12.2011* Time : *13:40* Duration : *120 minutes*	Last Name: Name : Department: Signature: Student No.: Section: 7 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1 (16)	2 (16)	3 (10)	4 (12)	5 (21)	6 (10)	7 (15)		

1. *(4×4pts)* Choose one corresponding differential equation from the list below for each of the following mechanical systems. Write your answers in the boxes provided.

- Mechanical Spring Systems:
  - **(i)** undamped, free, natural frequency $= 4$ $\boxed{\text{E}}$
  - **(ii)** overdamped, free $\boxed{\text{G}}$
  - **(iii)** undamped, forced, with resonance, natural frequency$= 3$ $\boxed{\text{C}}$
  - **(iv)** damped (not overdamped), forced, no resonance $\boxed{\text{F}}$
- Differential Equations List:
  - **(A)** $y'' - 16y = \cos(4t)$
  - **(B)** $y'' + y' + 2y = 0$
  - **(C)** $y'' + 9y = \sin(3t)$
  - **(D)** $y'' + 2y' + 3y = 0$
  - **(E)** $2y'' + 32y = 0$
  - **(F)** $y'' + y' + y = \sin(4t)$
  - **(G)** $y'' + 3y' + 2y = 0$
  - **(H)** $y'' + 2y' + y = \cos(3t)$

2. *(8+8pts)* The 8th order homogeneous, linear, constant coefficient differential equation

$$y^{(8)} - y^{(7)} - 5y^{(6)} + 11y^{(5)} - 22y''' + 24y'' - 8y' = 0$$

has characteristic equation

$$(r^2 - 2r + 2)(r-1)^3(r+2)^2 r = 0$$

(i) Write the **general solution** to the homogeneous differential equation

$$y^{(8)} - y^{(7)} - 5y^{(6)} + 11y^{(5)} - 22y''' + 24y'' - 8y' = 0$$

The roots of the characteristic equation are $0,\ 1^{③},\ -2^{②},\ 1 \mp i$

So, the general solution is

$$y(t) = C_1 + C_2 e^{x} + C_3 x \cdot e^{x} + C_4 x^2 e^{x} + C_5 e^{-2x} + C_6 x \cdot e^{-2x} + C_7 e^{x} \cdot \cos x + C_8 e^{x} \sin x$$

(ii) Write the FORM of the **particular solution** Y_P used in the method of undeterminate solutions to solve the non-homogeneous differential equation

$$y^{(8)} - y^{(7)} - 5y^{(6)} + 11y^{(5)} - 22y''' + 24y'' - 8y' = f(x)$$

where

$$f(x) = e^{x}\big(x \sin(x) + \cos(2x) + x\big) + xe^{-2x} + 1$$

(Do not plug into the differential equation to solve for the coefficients.)

$$f(x) = x \cdot e^{x} \sin x + e^{x} \cos(2x) + x \cdot e^{x} + x \cdot e^{-2x} + 1$$

$$Y_p(x) = x\left[(Ax+B)e^{x} \sin x + (Cx+D)e^{x} \cos x\right] + E e^{x} \cos(2x) + F e^{x} \sin(2x)$$

$$+ x^3(Gx+H)e^{x} + x^2(Kx+L)e^{-2x} + x \cdot M$$

Name: ID:

3. *(5+5pts)* Compute the following Laplace transforms:

(i) $\mathcal{L}\{e^{2t}\cos(t) + t\,u_1\}$ $= \mathcal{L}\{e^{2t}\cdot\cos t\} + \mathcal{L}\{u_1\cdot((t-1)+1)\}$

$$= \frac{(s-2)}{(s-2)^2+1^2} + e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

(ii) $\mathcal{L}\{e^{2t+1}\sin(3(t-4))\,u_4\}$ $= \mathcal{L}\{e^{2(t-4)+9}\sin(3(t-4))\,u_4\}$

$$= e^9\,\mathcal{L}\{e^{2(t-4)}\sin(3(t-4))\,u_4\}$$

$$= e^9\cdot e^{-4s}\cdot\frac{3}{(s-2)^2+3^2}$$

4. *(6+6pts)* Compute the following inverse Laplace transforms (without using convolutions):

(i) $\mathcal{L}^{-1}\left\{\dfrac{2s^3-2s+1}{s^2(s^2-1)}\right\}$

$$\frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-1} + \frac{d}{s+1} = \frac{2s^3-2s+1}{s^2(s^2-1)}$$

$s(s^2-1)$ (s^2-1) $s^2(s+1)$ $s^2(s-1)$

$as^3 - as + bs^2 - b + cs^3 + cs^2 + ds^3 - ds^2 = 2s^3 - 2s + 1 \Rightarrow$

$$\left.\begin{array}{l} a+c+d=2 \\ b+c-d=0 \\ -a=-2 \\ -b=1 \end{array}\right\} \begin{array}{l} a=2 \\ b=-1 \\ c=\frac{1}{2} \\ d=-\frac{1}{2} \end{array}$$

$$\mathcal{L}^{-1}\left\{2\cdot\frac{1}{s} - 1\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\} = 2 - t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

(ii) $\mathcal{L}^{-1}\left\{e^{-3s+1}\dfrac{-2s}{s^2+2s+5}\right\}$ $= \mathcal{L}^{-1}\left\{-e\cdot e^{-3s}\cdot\dfrac{2s}{s^2+2s+1+4}\right\}$

$$= -2e\,\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{s}{(s+1)^2+2^2}\right\}$$

$$= -2e\,\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{s+1-1}{(s+1)^2+2^2}\right\}$$

$$= -2e\cdot\left[\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{s+1}{(s+1)^2+2^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-3s}\frac{2}{(s+1)^2+2^2}\right\}\right]$$

$$= -2e\left(u_3(t)\cdot e^{-(t-3)}\cdot\cos(2(t-3)) - \frac{1}{2}\cdot u_3(t)\cdot e^{-(t-3)}\cdot\sin(2(t-3))\right)$$

5. *(21pts)* Solve the following initial value problem (without using convolutions):

$$y'' + y = \delta(t-4) + f(t), \qquad y(0) = 1, \quad y'(0) = 2 \qquad \text{where} \quad f(t) = \begin{cases} 1, & 0 \leqslant t < 1 \\ 2-t, & 1 \leqslant t < 2 \\ 0, & 2 \leqslant t \end{cases}$$

Here, $f(t) = 1 + u_1(t)\cdot(1-t) + u_2(t)\cdot(t-2)$

$= 1 - u_1(t)(t-1) + u_2(t)(t-2)$

Let's apply Laplace transform to the differential equation

$$s^2 Y(s) - s\cdot y(0) - y'(0) + Y(s) = e^{-4s} + \frac{1}{s} - e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\cdot\frac{1}{s^2}$$

$$(s^2+1)Y(s) - s - 2 = e^{-4s} + \frac{1}{s} - e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}$$

$$Y(s) = \frac{e^{-4s}}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-s}\cdot 1}{s^2(s^2+1)} + \frac{e^{-2s}\cdot 1}{s^2(s^2+1)} + \frac{s+2}{s^2+1}$$

$$\frac{1}{s\cdot(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \quad \text{and} \quad \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Hence,

$$Y(s) = e^{-4s}\cdot\frac{1}{s^2+1} + \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + e^{-2s}\cdot\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + \frac{s}{s^2+1} + 2\cdot\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = u_4(t)\cdot\sin(t-4) + 1 - \cancel{\cos(t)} - u_1(t)\cdot\big((t-1) - \sin(t-1)\big)$$

$$+ u_2(t)\cdot\big((t-2) - \sin(t-2)\big) + \cancel{\cos(t)} + 2\cdot\sin(t)$$

$$= 1 + 2\sin(t) - u_1(t)\big((t-1) - \sin(t-1)\big)$$
$$+ u_2(t)\big((t-2) - \sin(t-2)\big)$$
$$+ u_4(t)\sin(t-4)$$

6. *(5+5pts)* Compute the following convolutions using the indicated method.

(i) $t * e^{2t}$ (use the definition of convolution)

$$t * e^{2t} = \int_0^t e^{2(t-\tau)} \cdot \tau \, d\tau = e^{2t}\int_0^t e^{-2\tau} \cdot \tau \, d\tau = e^{2t}\left(-\frac{\tau e^{-2\tau}}{2}\Big|_0^t + \frac{1}{2}\int_0^t e^{-2\tau} d\tau \right)$$

$u = \tau \quad du = d\tau$

$dv = e^{-2\tau} d\tau \quad v = -\frac{e^{-2\tau}}{2}$

$$= e^{2t}\left(-\frac{\tau . e^{-2\tau}}{2}\Big|_0^t - \frac{1}{4} e^{-2\tau}\Big|_0^t \right)$$

$$= e^{2t}\left(\left(-t . \frac{e^{-2t}}{2} - 0\right) + \left(-\frac{1}{4} e^{-2t} + \frac{1}{4}\right)\right) = -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}$$

(ii) $u_2 * \delta(t-3)$ (use the convolution theorem)

By convolution theorem, $\mathcal{L}\{u_2 * \delta(t-3)\} = \frac{e^{-2s}}{s} \cdot e^{-3s} = e^{-5s} \cdot \frac{1}{s}$

$\mathcal{L}^{-1}\{e^{-5s} \cdot \frac{1}{s}\} = u_2 * \delta(t-3) = u_5$

7. *(15pts)* Find a function $g(t) \neq 0$ with the property that $\mathcal{L}\left\{\frac{d}{dt}(g * f)\right\} = \mathcal{L}\{f\}$ for **all** functions $f(t)$.
(To receive credit, you must show that your function has this property.)

$$\mathcal{L}\left\{\frac{d}{dt}(g*f)\right\} = s \cdot \mathcal{L}\{g*f\} - (g*f)(0) \underset{\text{Convolution Thm}}{=} s \cdot G(s) \cdot F(s) - (g*f)(0)$$

By definition, $(g*f)(t) = \int_0^t g(t-\tau)f(\tau)d\tau$, so $(g*f)(0) = 0$

We get, $\quad s \cdot G(s) \cdot F(s) = F(s)$

$$s \cdot G(s) \cdot F(s) - F(s) = 0$$

$$F(s)\,(s \cdot G(s) - 1) = 0 \qquad \text{for all } f(t) \text{ i.e for all } F(s)$$

Then $\quad G(s) = \frac{1}{s}$

Hence, $g(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = \underline{1}$

$\square$