METU - NCC

	DIFFERENTIAL EQUATIONS FINAL EXAM	
Code : *MAT 219* Acad.Year : *2011-2012* Semester : *Fall* Date : *15.1.2012* Time : *9:00* Duration : *150 minutes*	Last Name: Name : Department: Signature:	Student No.: Section:
	8 QUESTIONS ON 7 PAGES TOTAL 100 POINTS	

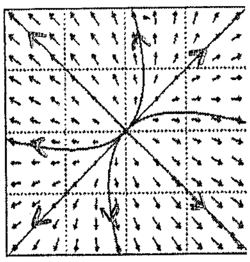
1 (8)	2 (15)	3 (15)	4 (12)	5 (5)	6 (10)	7 (10)	8 (15)		

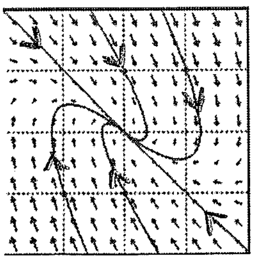
Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

Good luck!

1. *(8× 1pts)* The parts below give eigenvalues for a system of equations $\mathbf{x}' = A\mathbf{x}$. Match each set of eigenvalues with a possible phase portrait (write **one** phase portrait letter in each box).

- $\boxed{A}$ $\lambda_1 = 1,\ \lambda_2 = 2$
- $\boxed{D}$ $\lambda_1 = -1,\ \lambda_2 = 2$
- $\boxed{G}$ $\lambda_1 = -1,\ \lambda_2 = -2$
- $\boxed{I}$ $\lambda_1 = -1 + i,\ \lambda_2 = -1 - i$
- $\boxed{H}$ $\lambda_1 = i,\ \lambda_2 = -i$
- $\boxed{E}$ $\lambda_1 = 1,\ \lambda_2 = 1$
- $\boxed{J}$ $\lambda_1 = 0,\ \lambda_2 = 1$
- $\boxed{K}$ $\lambda_1 = 0,\ \lambda_2 = 0$


(A)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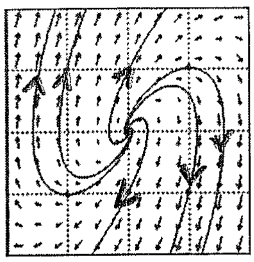
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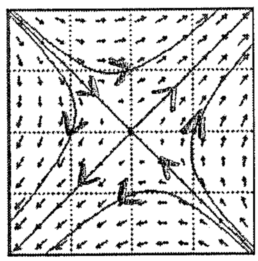
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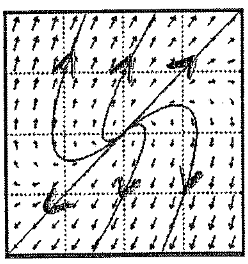
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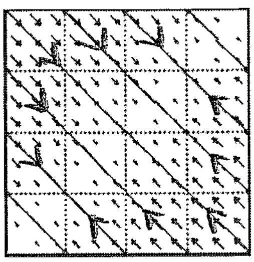
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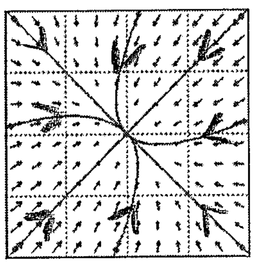
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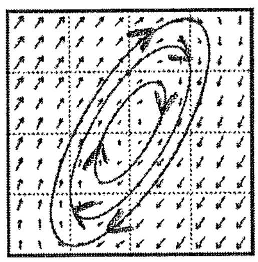
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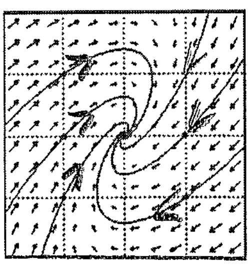
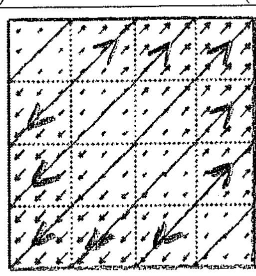
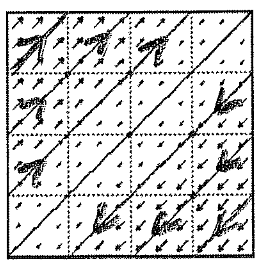

(I)


(J)


(K)

2. *(5+5+5pts)* Consider the system of differential equations $\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}}_{A} \mathbf{x}$

(a) Find the eigenvalues and eigenvectors of the matrix.

(Also find generalized eigenvectors if they exist.)

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 2 & 3-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda) \; ; \quad \boxed{\lambda_1 = 3,\ \lambda_2 = \lambda_3 = 2}$$

$\lambda = 3$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{array}{c} -u_1 = 0 \\ u_1 - u_2 = 0 \\ 2u_2 = 0 \end{array} \Rightarrow u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{array}{c} 0 = 0 \\ v_1 = 0 \\ 2v_2 + v_3 = 0 \end{array} \Rightarrow v = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad \begin{array}{c} 0 = 0 \\ w_1 = 1 \\ 2w_2 + w_3 = -2 \end{array} \Rightarrow \boxed{w = \begin{bmatrix} 1 \\ s \\ -2-2s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}$$

(b) Write the general solution to the system.

$$\boxed{x(t) = c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}e^{2t} + c_3\left(t\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)e^{2t}}$$

(c) Solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

$$X(0) = c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_3 \\ c_2 \\ c_1 - 2c_2 - 2c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{l} c_3 = 1 \\ c_2 = 1 \\ c_1 = 2 \end{array}$$

$$\boxed{X(t) = \begin{bmatrix} e^{2t} \\ (t+1)e^{2t} \\ 2e^{3t} - 4e^{2t} - 2te^{2t} \end{bmatrix}}$$

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3. *(5+10pts)* The matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$, with eigenvalues $\lambda_1 = 1+i$, $\lambda_2 = 1-i$, respectively.

(a) Write the general **real** solution to the equation $\mathbf{x}' = A\mathbf{x}$.

$$\boxed{x_h(t) = \left(c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}\right) e^t}$$

(b) Find the solution to the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} -e^t \sec(t) \\ e^t \end{bmatrix}$.

$$\psi = \begin{bmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{bmatrix} \Rightarrow \psi^{-1} = \frac{1}{e^{2t}} \begin{bmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{bmatrix}$$

$$u' = \psi^{-1} g = \begin{bmatrix} e^{-t} \cos t & -e^{-t} \sin t \\ e^{-t} \sin t & e^{-t} \cos t \end{bmatrix} \begin{bmatrix} -e^t \sec(t) \\ e^t \end{bmatrix} = \begin{bmatrix} -1 - \sin t \\ -\tan t + \cos t \end{bmatrix}$$

$$\Rightarrow u = \begin{bmatrix} \int (-1 - \sin t)\,dt \\ \int (-\tan t + \cos t)\,dt \end{bmatrix} = \begin{bmatrix} -t + \cos t \\ \ln|\cos t| + \sin t \end{bmatrix}$$

$$x_p = \psi u = \begin{bmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} -t + \cos t \\ \ln|\cos t| + \sin t \end{bmatrix}$$

$$x_p = \begin{bmatrix} -t \cos t\, e^t + e^t + (\ln|\cos t|) \sin t\, e^t \\ t \sin t\, e^t + (\ln|\cos t|) \cos t\, e^t \end{bmatrix}$$

$$X(t) = X_h + X_p$$

$$\boxed{X(t) = c_1 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} + c_2 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + \begin{bmatrix} -e^t\, t \cos t + e^t + e^t (\ln|\cos t|) \sin t \\ e^t\, t \sin t + e^t (\ln|\cos t|) \cos t \end{bmatrix}}$$

4. *(5+5+5+2 pts)* Consider the following initial value problem,

$$y'' - 2y' - 3y = 0, \qquad y(0) = 2, \quad y'(0) = 4.$$

(a) Solve the problem directly using its characteristic equation.

Char. Eqn: $r^2 - 2r - 3 = 0 \Rightarrow r_1 = -1;\ r_2 = 3$

$y(t) = c_1 e^{-t} + c_2 e^{3t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= -c_1 + 3c_2 = 4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \tfrac{1}{2} \\ c_2 &= \tfrac{3}{2} \end{aligned}$$

So, $\boxed{y(t) = \frac{1}{2} e^{-t} + \frac{3}{2} e^{3t}}$

(b) Solve the same problem using Laplace transforms.

After taking the Laplace transform:

$(s^2 Y - s.2 - 4) - 2(sY - 2) - 3Y = 0 \Rightarrow Y = \dfrac{2s}{s^2 - 2s - 3}$

$$Y = \frac{A}{s+1} + \frac{B}{s-3} \Rightarrow \left.\begin{aligned} A + B &= 2 \\ -3A + B &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{2} \\ B &= \tfrac{3}{2} \end{aligned}$$

$\Rightarrow \boxed{y(t) = \frac{1}{2} e^{-t} + \frac{3}{2} e^{3t}}$

(c) Convert the initial value problem on the previous page (including the initial values) to a system of linear differential equations by setting $x_1 = y$ and $x_2 = y'$. Write the system in matrix form and solve it (don't forget the initial values).

$$\left.\begin{array}{l} x_1 = y \Rightarrow x_1' = x_2 \\ x_2 = y' \Rightarrow x_2' = \underset{\uparrow \text{ using eqn.}}{2x_2 + 3x_1} \end{array}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

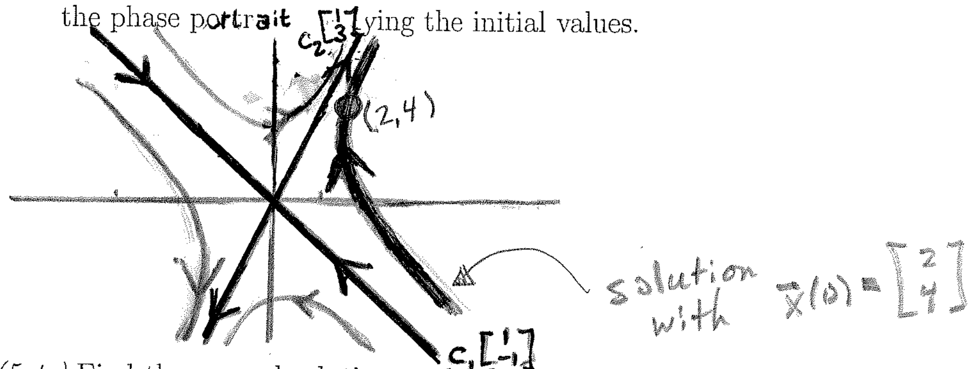
$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 \Rightarrow \lambda_1 = -1 ; \lambda_2 = 3$$

$\lambda_1 = -1$: $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 + u_2 = 0 \\ 3u_1 + 3u_2 = 0 \end{array} \Rightarrow u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 3$: $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -3v_1 + v_2 = 0 \\ 3v_1 - v_2 = 0 \end{array} \Rightarrow v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} \Rightarrow x(0) = \begin{bmatrix} c_1 + c_2 \\ -c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = \frac{3}{2} \end{array}$$

So, $\boxed{x(t) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \frac{3}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}}$

(d) Draw the phase portrait for the general solution from (c), and mark the solution in the phase portrait satisfying the initial values.



5. *(5pts)* Find the general solution to the first order differential equation

$$t \ln(t) \frac{dy}{dt} = te^t - y.$$

$$\frac{dy}{dt} + \frac{1}{t \ln t} y = \frac{e^t}{\ln t} \Rightarrow \mu = e^{\int \frac{1}{t \ln t} dt} = e^{\ln|\ln t|} = \ln t$$

$$\Rightarrow \underbrace{\ln t \frac{dy}{dt} + \frac{1}{t} y}_{} = e^t$$

$$(\ln t \cdot y)' = e^t \Rightarrow y \ln t = e^t + C$$

$$\boxed{y = \frac{e^t + C}{\ln t}}$$

6. *(10pts)* Use separation of variables to convert the following partial differential equation (with boundary conditions) into a pair of ordinary differential equations (with boundary conditions).

$$xe^y u_{yy} + y^2 u_{xx} = 0, \qquad u(0,y) = u(4,y) = 0, \quad u_y(x,0) = u_y(x,2) = 0.$$

Let $u(x,y) = X(x)\,Y(y)$ then eqn and initial conditions becomes;

$xe^y X Y'' + y^2 X'' Y = 0$

$-\dfrac{e^y Y''}{y^2 Y} = \dfrac{X''}{xX} = -\lambda$

- $X(0)Y(y) = X(4)Y(y) = 0$

  $\Rightarrow X(0) = X(4) = 0$

- $X(x)Y'(0) = X(x)Y'(2) = 0$

  $\Rightarrow Y'(0) = Y'(2) = 0.$

So, we have two initial value problems!

$$\boxed{\begin{array}{c} X'' + \lambda x \cdot X = 0 \\ X(0) = X(4) = 0 \end{array}} \quad \text{and} \quad \boxed{\begin{array}{c} e^y Y'' - \lambda y^2 Y = 0 \\ Y'(0) = Y'(2) = 0 \end{array}}$$

7. *(10pts)* For the following boundary value problem, find λ so that the solution is nonzero, and give the solution.

$$y'' + 2y' + \lambda y = 0, \quad \boxed{(\lambda > 1)} \quad y(0) = 0, \quad y(2) = 0.$$

Char. Eqn: $r^2 + 2r + \lambda = 0 \Rightarrow r = \dfrac{-2 \mp \sqrt{4 - 4\lambda}}{2} = -1 \mp \sqrt{1-\lambda}$

~~if $\lambda = 1$ then $y = c_1 e^{-t} + c_2 t e^{-t}$ $\Rightarrow$ $y(0) = c_1 = 0$, $y(2) = c_2 2 e^{-2} = 0$ $\Rightarrow c_1 = c_2 = 0 \Rightarrow y = 0$~~

~~if $\lambda < 1$ then $y = c_1 e^{(-1-\sqrt{1-\lambda})t} + c_2 e^{(-1+\sqrt{1-\lambda})t}$ $\Rightarrow$ $y(0) = c_1 + c_2 = 0$, $y(2) = c_1 e^{(-1-\sqrt{1-\lambda})2} + c_2 e^{(-1+\sqrt{1-\lambda})2} = 0$ $\Rightarrow c_1 = c_2 = 0 \Rightarrow y = 0.$~~

if $\lambda > 1$ then $y = e^{-t}(c_1 \cos(\sqrt{\lambda-1}\,t) + c_2 \sin(\sqrt{\lambda-1}\,t))$ $\Rightarrow y(0) = c_1 = 0$

$y(2) = c_2 e^{-2} \sin(2\sqrt{\lambda-1}) = 0$

$\Rightarrow 2\sqrt{\lambda-1} = k\pi$

So, $$\boxed{y(t) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi}{2} t\right) e^{-t} \quad \text{and} \quad \lambda = \left(\frac{k\pi}{2}\right)^2 + 1}$$

8. *(15+5pts)* Compute the Fourier series of the **even** function $f(x) = \begin{cases} 2+x & -2 \leqslant x < -1 \\ 1+x & -1 \leqslant x < 0 \\ 1-x & 0 \leqslant x < 1 \\ 2-x & 1 \leqslant x < 2 \end{cases}$

Write the **first four nonzero terms** of the Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{where } L = 2.$$

Since f is even, $b_n = 0$ and we only need to calculate a_n's.

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = 1$$

$$a_n = \frac{2}{2}\int_0^2 f(x).\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

Remember $\int x.\cos\left(\frac{n\pi x}{2}\right)dx = \frac{x\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \int \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}dx = \frac{x\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$

So, $a_n = \left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \left(\frac{x\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^1 + \left[\frac{2\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \left(\frac{x\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_1^2$.

$$a_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \cancel{\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}} - \cancel{\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}} + \frac{1}{\left(\frac{n\pi}{2}\right)^2} - \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} - \frac{2\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} + \cancel{\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}} + \cancel{\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}}$$

$$a_n = \frac{1-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} - \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}$$

So,

$$\boxed{f(x) = \frac{1}{2} + \left(\frac{1-\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2}\right)\cos\left(\frac{\pi x}{2}\right) + \frac{2}{\pi^2}\cos(\pi x) + \left(\frac{1+\frac{3\pi}{2}}{\left(\frac{3\pi}{2}\right)^2}\right)\cos\left(\frac{3\pi x}{2}\right) + \cdots}$$

In the space below, graph the **Fourier series** (not f) for $-6 \leqslant t \leqslant 6$.
(For full credit you must indicate the correct values at discontinuities.)