M E T U
Northern Cyprus Campus

	Applied Mathematics for Engineers Final Exam	

Code : *Math 210* Acad.Year: *2012-2013* Semester : *Spring* Date : *06.6.2013* Time : *16:00* Duration : *150 minutes*	Last Name: Name : Department: Signature: Student No: Section: 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (14)	2 (20)	3 (10)	4 (6)	5 (16)	6 (16)	7 (18)	Bonus		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1. (10+4) Consider the differential equation $u'' + 2u' + 3u = x \qquad u(0) = 0 \quad u(4) = 0$

(a) For $h = 1$, write the difference equations for the differential equation in the matrix form.

Fixed - Fixed Differential Eqn.

$U(0) = U_0 = 0\ ,\ U(4) = U_4 = 0$

0 1 2 3 4

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

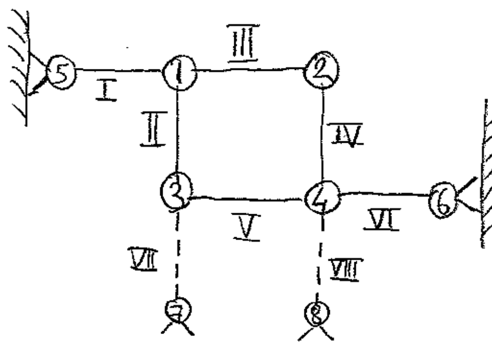
$$\frac{1}{1^2}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + 2\cdot\frac{1}{2\cdot 1}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) Verify that $u = \begin{bmatrix} 0 \\ 9 \\ -4 \\ 3 \\ 0 \end{bmatrix}$ is a discrete solution to the above differential equation in (a).

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \checkmark$$

2. (4+4+4+4+4) Given the following truss



(a) Find the elongation matrix A. i.e. $e = Au$.

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{bmatrix}$$

(b) Explain why the truss is unstable.

After the row operation $R_1 + R_3$, A becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 6 < 8 = \#$ of columns so it's unstable.

Also, we see u_3^V and u_4^V are free varia

(c) Find a basis for the null space of A.

$A \cdot u = 0$

$$\begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{bmatrix} = u_3^V \overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}^{1^{st}} + u_4^V \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}^{2^{nd}}$$

$-u_4^H = 0 \Rightarrow u_4^H = 0$

$-u_3^H + u_4^H = 0 \Rightarrow u_3^H = 0$

$u_2^V - u_4^V = 0 \Rightarrow u_2^V = u_4^V$

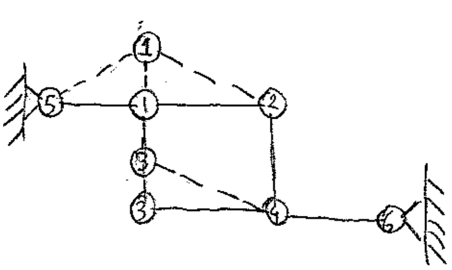
$u_2^H = 0$

$u_1^V - u_3^V = 0 \Rightarrow u_1^V = u_3^V$

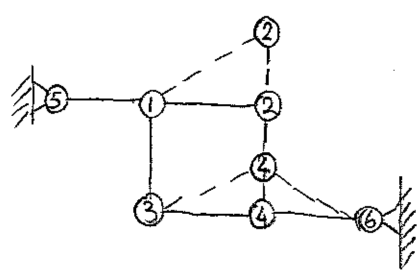
$u_1^H = 0$

(d) Draw the mechanism(s) corresponding to the basis element(s) of the null space of A.

1^{st} Mechanism



2^{nd} Mechanism



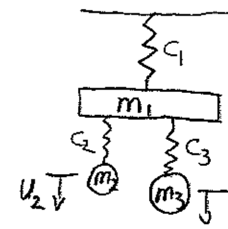
(e) By adding the least number of bars, make the truss stable. Prove the stability of the new truss. (Draw the bar(s) on the actual truss.) If we add bars VII and VIII, and apply $R_1 + R_3$, $R_6 \leftrightarrow R_7$, A becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank = 8 = # of columns

So, it's stable

3. (10pts) Consider the spring-mass system in the figure with three masses $m_1 = 5, m_2 = 1, m_3 = 3$ and three identical springs with spring constant equal to 1. If the system is in equilibrium, find the displacement of each mass.



$e_1 = u_1$

$e_2 = u_2 - u_1$

$e_3 = u_3 - u_1$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$K = A^T \cdot C \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

We need to solve $K \cdot u = f$

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 5g \\ -1 & 1 & 0 & g \\ -1 & 0 & 1 & 3g \end{array}\right] \xrightarrow[+3R_3+R_1]{-R_3+R_2} \left[\begin{array}{ccc|c} 0 & -1 & 2 & 14g \\ 0 & 1 & -1 & -2g \\ -1 & 0 & 1 & 3g \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{R_2+R_1} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 3g \\ 0 & 1 & -1 & -2g \\ 0 & 0 & 1 & 12g \end{array}\right]$$

$u_1 = 9$

$u_2 = 10g$

$u_3 = 12g$

$\lambda^3 = 9\lambda \Rightarrow \lambda(\lambda^2 - 9) = 0$

4. (1pt each) (a) A square matrix A satisfies the equation $A^3 = 9A$, its eigenvalues are 0, −3, +3.

(b) In this course (and mechanics in general), equations of the type $Ku = f$ appear in Equilibrium type problems, whereas eigenvalue problems $Ku = \lambda u$ appear in Oscillation type problems.

(c) In LU-decomposition of a square matrix A, the diagonal of the lower-triangular matrix L are 1's and the diagonal of the upper-triangular matrix U are pivots.

(d) Indicate a subject/a type of problem in which symmetric matrices appear: Equilibrium Spring-Mass System etc.

(Projection) Least Square Method)

5. (4+4+4+4)Given $f(x) = \begin{cases} -\frac{2}{\pi}x + 2, & 0 \le x \le \pi \\ \frac{2}{\pi}x - 2, & \pi \le x \le 2\pi \end{cases}$ and $g(x) = \sin(x) - \cos(2x)$

(a) For $N = 4$, discretize $f(x)$ and $g(x)$ on $[0, 2\pi]$ to get $\vec{f}$ and $\vec{g}$.

$0 \quad \pi/2 \quad \pi \quad 3\pi/2 \quad 2\pi$

$$f = \begin{bmatrix} f(0) \\ f(\pi/2) \\ f(\pi) \\ f(3\pi/2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad g = \begin{bmatrix} g(0) \\ g(\pi/2) \\ g(\pi) \\ g(3\pi/2) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

(b) Find **DFT** of $\vec{f}$ and $\vec{g}$. We need to multiply with $\frac{1}{4}\overline{F_4}$

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\frac{1}{4}\begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}}_{c} \qquad \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \underbrace{\frac{1}{4}\begin{bmatrix} 0 \\ -2i \\ -4 \\ 2i \end{bmatrix}}$$

(c) Compute **DFT** of $\overrightarrow{f \cdot g}$. $\overrightarrow{f \cdot g}$ will have $c \circledast d$ as DF coefficients

$$\begin{array}{rrrrrrrr} & & & & 2 & 0 & 2 & 4 \\ \times & & & & 2i & -4 & -2i & 0 \\ \hline & & & & 0 & 0 & 0 & 0 \\ & & & -4i & 0 & -4i & -8i & -4i \\ & & -8 & 0 & -8 & -16 & -8 & 0 \\ + & 4i & 0 & 4i & 8i & 4i & 0 & 4i \\ \hline & & & & -8+8i & -16 & -8-8i & 0 \end{array}$$

$$c \circledast d = \frac{1}{16}\begin{bmatrix} 0 \\ -8-8i \\ -16 \\ -8+8i \end{bmatrix}$$

(d) Compute **DFT** of $\vec{f} \circledast \vec{g}$. $\vec{f} \circledast \vec{g}$ will have $c \cdot d$ as D.F coefficients

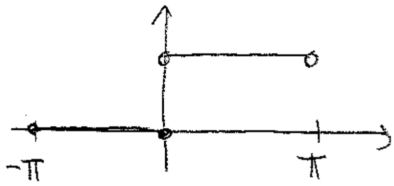
componentwise

$$\frac{1}{4}\begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 0 \\ -2i \\ -4 \\ 2i \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0 \\ -4i \\ 0 \\ 4i \end{bmatrix}$$

componentwise

6. (6+5+5) Functions $f(x), g(x)$ and $h(x)$ in this problem are all 2π-periodic. In each ~~case~~, find the complex Fourier series for the given function.

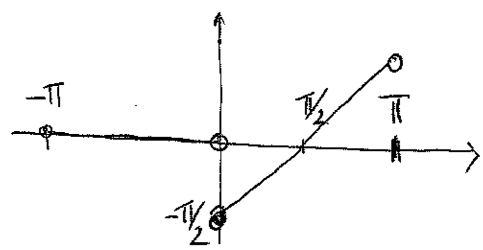
(a) $f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x < \pi \end{cases}$.



$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-ikx}\,dx = \frac{1}{2\pi}\left(\frac{e^{-ikx}}{-ik}\Bigg|_0^{\pi}\right)$$

$$= \frac{1}{2\pi i k}\left(1 - e^{-ik\pi}\right) = \frac{1}{2\pi i k}\left(1-(-1)^k\right)$$

(b) $g(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ x - \frac{\pi}{2} & \text{if } 0 \le x < \pi \end{cases}$.



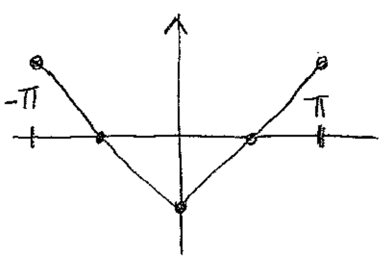
$g'(x) = f(x)$ i.e. $g(x) = \int f(x)\,dx$

$$d_k = \frac{C_k}{ik} = \frac{1}{2\pi i^2 k^2}\left(1-(-1)^k\right) = \frac{-1}{2\pi k^2}\left(1-(-1)^k\right)$$

$k \neq 0$

$$d_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = 0$$

(c) $h(x) = \begin{cases} -x - \frac{\pi}{2} & \text{if } -\pi \le x < 0 \\ x - \frac{\pi}{2} & \text{if } 0 \le x < \pi \end{cases}$.



$h(x) = g(x) + g(-x)$

$g(-x)$ will have F. coefficients $\tilde{d}_k = d_{-k}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(-x)\cdot e^{-ikx}\,dx = \frac{-1}{2\pi}\int_{\pi}^{-\pi} g(u) e^{iku}\,du$$

$u = -x$
$du = -dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(u) e^{iku}\,du = d$$

So $e_k = d_k + d_{-k}$

7. (3pts each) This problem has unrelated parts.

(a) Compute the complex Fourier Series of function $f(x) = \pi$.

$f(x) = \pi \qquad C_0 = \pi \quad C_k = 0$ for all $k \in \mathbb{Z}$ $k \neq 0$.

(b) Compute the complex Fourier coefficients c_0, c_1, c_{-1} of $g(x) = \cos(2x)\sin(3x)$

$\cos(2x) \cdot \sin(3x) = \frac{e^{i2x} + e^{-i2x}}{2} \cdot \frac{e^{i3x} - e^{-i3x}}{2i} = +\frac{1}{4i} e^{ix} - \frac{1}{4i} e^{-ix} + \frac{1}{4i} e^{5x} - \frac{1}{4i} e^{-5x}$

$C_0 = 0, \quad C_1 = \frac{1}{4i}, \quad C_{-1} = -\frac{1}{4i}$

(c) Compute the real Fourier Series of $h(x) = \cos^2(x)$.

$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$a_0 = \frac{1}{2}, \quad a_2 = \frac{1}{2}$ all others are 0.

(d) Compute the real Fourier Series of $k(x) = \sin(x - \frac{\pi}{2})$.

$\sin(x - \frac{\pi}{2}) = -\cos x$ so $a_1 = -1$ all others are 0.

(e) Find a function on $[-\pi, \pi]$, which is orthogonal to $\delta(x)$ with respect to the complex inner product $< f, g >_{\mathbb{C}} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$

$0 = \langle f, \delta(x) \rangle_{\mathbb{C}} = \int_{-\pi}^{\pi} f(x) \overline{\delta(x)}\, dx = \int_{-\pi}^{\pi} f(x) \cdot \delta(x)\, dx = f(0).$

$\delta(x)$ is real

Hence, any $f(x)$ on $[-\pi, \pi]$ with $f(0) = 0$.

(f) For $N = 3$, find a vector c such that $c \circledast c = c^2$ (componentwise product)

$c = [0\ 0\ 1]$

$$\begin{array}{r} 0\ 0\ 1 \\ \times\ 0\ 0\ 1 \\ \hline 0\ 0\ 1 \\ 0\ 0\ 0 \\ +\ 0\ 0\ 0 \\ \hline 0\ 0\ 1 \end{array}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Bonus(5pts) Produce a valid MatLab code for the following.

(a) 100x100 Identity matrix.

```
>> eye(100)
```

(b) 100x100 Centered First Difference matrix with $h = 0.01$.

```
>> C = toeplitz([0 -1 zeros(1,98)],[0 1 zeros(1,98)]);
>> C = 1/(2*0.01) * C
```

(c) Plot $\sin(x) + x^2$ from 0 to 2π

```
>> x = linspace(0,2π,100);
>> y = sin(x);
>> plot(x,y)
```