METU - NCC

Applied Mathematics for Engineers FINAL EXAM	
Code : *MAT 210* Acad.Year: *2011-2012* Semester : *SPRING* Date : *03.06.12* Time : 16:00 Duration : *150 minutes*	Last Name: ~~SOLUTIONS~~ Student No.: Name : Department: Section: Signature:
	8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (8)	2 (12)	3 (12)	4 (15)	5 (11)	6 (20)	7 (12)	8 (10)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

Good luck!

1. *(8 pts)* Write a 3×3 matrix equation discretizing the differential equation

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} = x^2 \qquad u(0) = 0, \quad u(4) = 0$$

$\Delta x = h = 1$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array} \qquad x^2 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

$$u'' = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = \frac{u_{i-1} - 2u_i + u_{i+1}}{1}$$

$$u' = \frac{-u(x_{i-1}) + u(x_{i+1})}{2 \cdot h} = \frac{-u_{i-1} + u_{i+1}}{2}$$

$$\overbrace{\frac{1}{1^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}^{-\frac{d^2u}{dx^2}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \overbrace{\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}^{du/dx} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1/2 & 0 \\ -3/2 & 2 & -1/2 \\ 0 & -3/2 & 2 \end{bmatrix} \bar{u} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

2. *(3×4 pts)* Show that $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is **not** positive definite using the following methods:

a) upper (left) determinants (e.g. principal minors)

$\det [1] = 1$

$\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 1 - 4 = -3$

$-3 < 0 \Rightarrow$ NOT positive definite.

b) pivots

LU decomposition $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$ $-3 < 0 \Rightarrow$ NOT positive definite,

c) eigenvalues

$0 = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

$\lambda = 3, -1$ $-1 < 0 \Rightarrow$ NOT positive definite.

d) the energy function

$[x \;\; y]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x \;\; y]\begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix} = x^2 + 4xy + y^2$

if $\begin{cases} x = 1 \\ y = -1 \end{cases}$ then $x^2 + 4xy + y^2 = 1 - 4 + 1 = -2 < 0$

NOT positive definite.

3. *(4×3 pts)* The following parts involve projection onto the line $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$.

a) Find the **projection matrix** for the line.
(i.e. Find the matrix P so that $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto the line.)

$P = A(A^TA)^{-1}A^T$ where $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\left([14]\right)^{-1}[1 \;\; 2 \;\; 3] = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

b) Compute the distance from $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ to the line.

distance $= |x - Px| = \left|\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 13 \\ 26 \\ 39 \end{bmatrix}\right| = \frac{1}{14}\left|\begin{bmatrix} 15 \\ -12 \\ 3 \end{bmatrix}\right|$

$= \frac{1}{14}\sqrt{15^2 + 12^2 + 3^2}$

c) Write one eigenvalue and eigenvector of the matrix P.
(Hint: no computation is required.)

Projection matrices have eigenvalues 1 & 0

1-eigenspace = stuff projected onto

0-eigenspace = $\perp$ stuff

EX: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a 1-eigenvector

EX: $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is a 0-eigenvector

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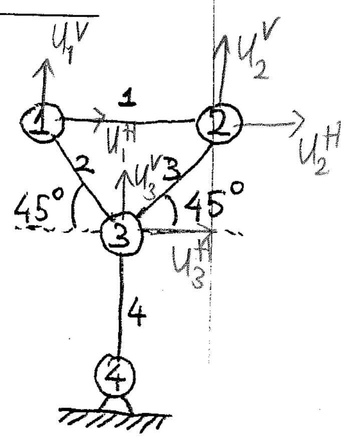
4. *(5×3 pts)* Complete the following for the truss system to the right:



a) Write the elongation matrix A.

$e_1 = -u_1^H\cos(0) + u_1^V\sin(0) + u_2^H\cos(0) - u_2^V\sin(0)$

$e_2 = -u_1^H\cos(45°) + u_1^V\sin(45°) + u_3^H\cos(45°) - u_3^V\sin(45°)$

$e_3 = u_2^H\cos(45°) + u_2^V\sin(45°) - u_3^H\cos(45°) - u_3^V\sin(45°)$

$e_4 = u_3^H\cos(90°) + u_3^V\sin(90°)$

b) Find the nullspace of A.

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0 & 0 & \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \\ 0 & 0 & \frac{\sqrt2}{2} & \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\frac{2}{\sqrt2}R_3]{-\frac{2}{\sqrt2}R_2} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

u_2^V & u_3^H are free

$u_3^V = 0$

$u_2^H = -u_2^V + u_3^H$

$u_1^V = u_2^H - u_3^H = -u_2^V$

$u_1^H = u_2^H = -u_2^V + u_3^H$

$u_2^V = u_2^V$

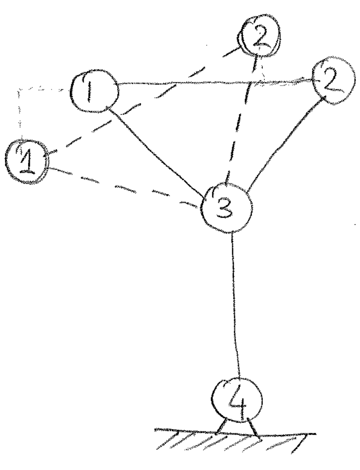
$u_3^H = u_3^H$

$$\begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \end{bmatrix} = u_2^V \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u_3^H \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

1^{st} 2^{nd}

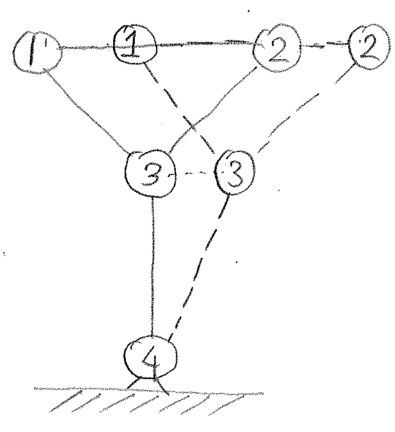
Mechanisms

c) Draw the mechanisms corresponding the the nullspace elements from (b).



1^{st} Mechanism

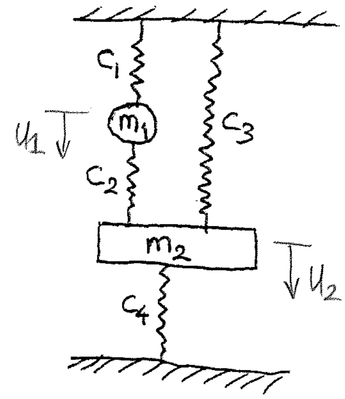
(Rotation around node 3)



2^{nd} Mechanism

(Rotation around node 4)

5. *(4+4+3 pts)* The following parts involve the spring system:



a) Write the elongation matrix for the spring system.

$e_1 = u_1$
$e_2 = u_2 - u_1$
$e_3 = u_2$
$e_4 = -u_2$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

b) Write the stiffness matrix for the spring system.

$$K = A^T \cdot C \cdot A$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 + c_4 \end{bmatrix}$$

c) Are eigenvalues used to solve equilibrium problems or oscillation problems?

Oscillation Problems: Square root of eigenvalues of $M^{-1}K$ gives us natural frequencies.

6. *(5×4 pts)* Let $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ with integral $F(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

a) Compute the coefficients c_n of the continuous exponential Fourier series for $f(x)$.

- $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \cdot e^{-inx}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\cdot e^{-inx}\,dx\right)$

$= \frac{1}{2\pi}\left(\frac{1}{-in} e^{-inx}\Big|_0^{\pi}\right) = \frac{i}{2\pi n}\left(e^{-in\pi} - 1\right)$ (Note: $-\frac{1}{i} = i$)

$= \frac{i}{2\pi n}\left((-1)^n - 1\right)$ (Note: $e^{-i\pi} = (-1)$)

- $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx = 1/2$

b) Use your answer from a) to get the coefficients for $F(x) = \int f(x)\,dx$.
(Be careful with c_0! It must be computed separately.)

Recall: $\mathcal{F}\{\int f\} = \frac{1}{-in}\mathcal{F}\{f\}$ for $n \neq 0$ (Just like Laplace...)

- $c_n = \frac{1}{-in}\left(\frac{i}{2\pi n}\left(e^{-in\pi} - 1\right)\right) = \frac{1}{2\pi n^2}\left(1 - e^{-in\pi}\right)$

$= \frac{1}{2\pi n^2}\left(1 - (-1)^n\right)$

- $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx = \pi/4$

c) Use your answer from b) to get the coefficients for $F(x - \pi)$.

Recall: $\mathcal{F}\{f(x-a)\} = e^{-ian}\mathcal{F}\{f\}$ (Just like Laplace...)

- $c_n = \frac{1}{2\pi n^2} e^{-in\pi}\left(1 - e^{-in\pi}\right)$

- same c_0

$= \frac{1}{2\pi n^2}\left(e^{-in\pi} - 1\right)$ (Note: $e^{-i2\pi} = 1$, so $e^{-i2\pi n} = 1^n$)

$= \frac{1}{2\pi n^2}\left((-1)^n - 1\right)$

d) Use your answers from all of the parts above to compute the coefficients for the sawtooth wave $S(x) = F(x - \pi) + f(x) - F(x)$.
(No credit will be given for direct computation.)

(Note: Sawtooth wave is actually $F(x-\pi) + \pi f(x) - F(x)$)

- $c_n = \frac{1}{2\pi n^2}\left((-1)^n - 1\right) + \frac{i}{2\pi n}\left((-1)^n - 1\right) - \frac{1}{2\pi n^2}\left(1 - (-1)^n\right)$

$= \frac{1}{\pi n^2}\left((-1)^n - 1\right) + \frac{i}{2\pi n}\left((-1)^n - 1\right)$

$= \left(\frac{1}{n} + \frac{i}{2}\right)\frac{1}{\pi n}\left((-1)^n - 1\right)$

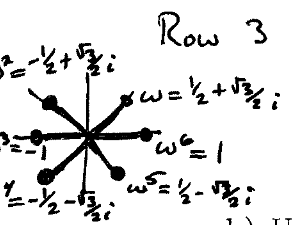
(Some students may have written answer w/ $e^{-in\pi}$ and $e^{-2in\pi}$... also ok.)

- $c_0 = \pi/4 + 1/2 - \pi/4 = 1/2$

7. *(6+6 pts)* The parts below $f = [1,\ 0,\ 1,\ -1,\ 1,\ 0]^T$ is a discrete signal with length 6.

$$\left(\omega = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}} = \tfrac12 + \tfrac{\sqrt3}{2} i \right)$$

a) Compute (**only**) the second discrete Fourier coefficient, c_2 for f.



Row 3 of F_6 is $[1\ \ \omega^2\ \ \omega^4\ \ \omega^6\ \ \omega^8\ \ \omega^{10}] = [1\ \ \omega^2\ \ \omega^4\ \ 1\ \ \omega^2\ \ \omega^4]$

$$= [1\ \ (-\tfrac12+\tfrac{\sqrt3}{2}i)\ \ (-\tfrac12-\tfrac{\sqrt3}{2}i)\ \ 1\ \ (-\tfrac12+\tfrac{\sqrt3}{2}i)\ \ (-\tfrac12-\tfrac{\sqrt3}{2}i)]$$

$$c_2 = \tfrac16\left(1\cdot 1 + \cancel{0\cdot(-\tfrac12-\tfrac{\sqrt3}{2}i)} + 1\cdot(-\tfrac12+\tfrac{\sqrt3}{2}i) - 1\cdot(1) + 1\cdot(-\tfrac12-\tfrac{\sqrt3}{2}i) + \cancel{0(-\tfrac12+\tfrac{\sqrt3}{2}i)}\right)$$

$$= \tfrac16(-1) = \boxed{-\tfrac16}$$

b) Use your answer from a) to get c_4.

$$c_4 = \overline{c_{6-4}} = \overline{c_2} = \overline{-\tfrac16} = \boxed{-\tfrac16}$$

8. *(5+5 pts)* In the parts below $g = [g_0,\ g_1,\ g_2,\ g_3]$ is a discrete signal with Fourier transform $c = [1,\ i+1,\ -2,\ -i+1]^T$.

a) What is the Fourier transform of g^2?

Fourier transform of g^2 is given by cyclic convolution:

$$\begin{array}{rcccc}
 & (-i+1) & -2 & (i+1) & 1 \\
\circledast & (-i+1) & -2 & (i+1) & 1 \\
\hline
 & (-i+1) & -2 & (i+1) & 1 \\
 & -2(i+1) & 2i & (i+1) & 2 \\
 & -2(i+1) & -2 & -2(-i+1) & 4 \\
+ & (-i+1) & -2i & -2(-i+1) & 2 \\
\hline
 & (-6i-2) & -4 & (6i-2) & 9
\end{array}$$

$\Longrightarrow \boxed{[9\ \ (6i-2)\ \ -4\ \ (-6i-2)]}$

Scratch work:

$(i+1)^2 = 2i - 1 + 1 = 2i$

$(i+1)(-i+1) = 1 + 1 = 2$

$(-i+1)^2 = -1 - 2i + 1 = -2i$

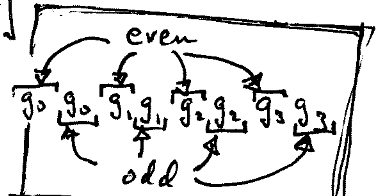
b) Use FFT to compute the Fourier transform of $h = [g_0,\ g_0,\ g_1,\ g_1,\ g_2,\ g_2,\ g_3,\ g_3]$. (Remember that the inverse of DFT matrix F_N is equal to $\frac{1}{N}\overline{F_N}$).

<u>Recall</u>: Fast Fourier Transform $\Longleftrightarrow$ factorization of F_8:

$$F_8 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$$

Fourier transform of h $= \frac18 \overline{F_8}\, h = \frac12 \begin{bmatrix} I & \overline{D} \\ I & -\overline{D} \end{bmatrix} \begin{bmatrix} \frac14\overline{F_4} & 0 \\ 0 & \frac14\overline{F_4} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix} h$

$$= \frac12 \begin{bmatrix} I & \overline{D} \\ I & -\overline{D} \end{bmatrix} \begin{bmatrix} \frac14\overline{F_4} & 0 \\ 0 & \frac14\overline{F_4} \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}$$



$$= \frac12 \begin{bmatrix} I & \overline{D} \\ I & -\overline{D} \end{bmatrix} \begin{bmatrix} 1 \\ i+1 \\ -2 \\ -i+1 \\ 1 \\ i+1 \\ -2 \\ -i+1 \end{bmatrix}$$

$\frac14\overline{F_4}\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}$ is Fourier transf. of g

Recall $D = \begin{bmatrix} 1 & & & 0 \\ & \omega & & \\ & & \omega^2 & \\ 0 & & & \omega^3 \end{bmatrix}$

$\omega = e^{2\pi i/8} = \frac{1}{\sqrt2}(1+i)$

$\overline{\omega} = \frac{1}{\sqrt2}(1-i)$

$\overline{\omega}^2 = -i$

$\overline{\omega}^3 = \frac{1}{\sqrt2}(-1-i)$

$$\overline{D}\begin{bmatrix} 1 \\ i+1 \\ -2 \\ -i+1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 \\ \frac{1}{\sqrt2}(1-i)\cdot(i+1) \\ -i\cdot(-2) \\ \frac{1}{\sqrt2}(-1-i)\cdot(-i+1) \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt2 \\ 2i \\ -\sqrt2 \end{bmatrix}$$

$$= \frac12 \begin{bmatrix} 1+1 \\ i+1+\sqrt2 \\ -2+2i \\ -i+1+(-\sqrt2) \\ 1-1 \\ i+1-\sqrt2 \\ -2-2i \\ \vdots \end{bmatrix}$$