M E T U
Northern Cyprus Campus

	Applied Mathematics for Engineers Midterm	
Code : *Math 210* Acad.Year : *2010-2011* Semester : *Spring* Date : *23.4.2011* Time : *10:00* Duration : *120 minutes*	Last Name: Name : Department: Signature:	Student No: Section:
	5 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1. (10+10) For the differential equation

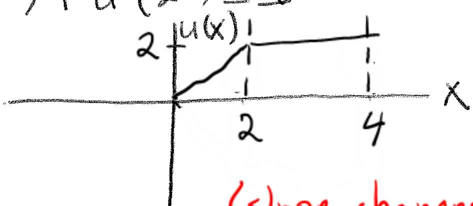
$$-u''(x) = \delta(x-2) \qquad u(0) = 0, u'(4) = 0$$

(a) Find the continuous solution. Graph your answer.

$0 < x < 2 \Rightarrow u'' = 0 \Rightarrow u(x) = Ax + B$

$2 < x < 4 \Rightarrow u'' = 0 \Rightarrow u(x) = Cx + D$

$$\left.\begin{array}{l} 0 = u(0) = B \\ 0 = u'(4) = C \\ u(2^-) = u(2^+) \Rightarrow 2A + B = 2C + D \\ u'(2^+) + u'(2^-) = 1 \Rightarrow A + C = 1 \end{array}\right\} \begin{array}{l} A = 1, \ B = 0 \\ C = 0, \ D = 2 \end{array}$$



$$\boxed{u(x) = \begin{cases} x & \text{if } 0 \le x < 2 \\ 2 & \text{if } 2 \le x \le 4 \end{cases}}$$

(slope changes by -1 at $x = 2$)

(b) For n=3, write the difference equations for the same differential equation, and put the equations in matrix form. (DO NOT SOLVE THE EQUATION)

$h = \dfrac{4-0}{4} = 1$

$u_0 = 0$ since $u(0) = 0$

$-u_0 + 2u_1 - u_2 = 0$

$-u_1 + 2u_2 - u_3 = 0$

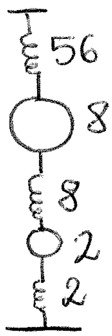
$-u_2 + 2u_3 - u_4 = 0$

$u_3 - u_4 = 0$

$$\boxed{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

2. (4+4+6+10) Consider the spring-mass system with two masses $m_1 = 8, m_2 = 2$ and three springs with spring constants $c_1 = 56, c_2 = 8, c_3 = 2$. Both ends are fixed. Assume that there is no friction and external force acting, and the masses are moving up and down. Let $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ be the displacements of the masses at any time t

(a) Write the elongations e_1, e_2, e_3 of the springs in terms of u_1 and u_2.



$$e_1 = u_1$$
$$e_2 = u_2 - u_1$$
$$e_3 = -u_2$$

$$\boxed{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

(b) Find the stiffness matrix K.

$$K = A^T C A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 56 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 56 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 56 & -8 & 0 \\ 0 & 8 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 64 & -8 \\ -8 & 10 \end{bmatrix}}$$

(c) Calculate the frequencies ω_1, ω_2.

$$M^{-1}K = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 64 & -8 \\ -8 & 10 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -4 & 5 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 8 & 1 \\ 4 & \lambda - 5 \end{vmatrix} = 0 \Rightarrow (\lambda - 8)(\lambda - 5) - 4 = 0$$
$$\lambda^2 - 13\lambda + 36 = 0$$
$$(\lambda - 9)(\lambda - 4) = 0$$
$$\lambda_1 = 9, \quad \lambda_2 = 4$$

$$\boxed{\omega_1 = \sqrt{\lambda_1} = 3} \qquad \boxed{\omega_2 = \sqrt{\lambda_2} = 2}$$

eigenvectors for λ_1:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigenvectors for λ_2

$$\left[\begin{array}{cc|c} -4 & 1 & 0 \\ 4 & -1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(d) Find the displacements of the masses $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ at any time t if $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} u_1'(0) \\ u_2'(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

$$u(t) = (A\cos 3t + B\sin 3t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (C\cos 2t + D\sin 2t)\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$u(0) = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 5 & 5 \end{array}\right] \qquad \boxed{\begin{array}{l} C = 1 \\ A = 2 \end{array}}$$

$$u'(t) = (-3A\sin 3t + 3B\cos(3t))\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (-2C\sin 2t + 2D\cos 2t)\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$u'(0) = 3B\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2D\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 5 \\ -3 & 8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 5 \\ 0 & 10 & 5 \end{array}\right] \qquad \boxed{\begin{array}{l} D = 1/2 \\ B = 4/3 \end{array}}$$

$$\boxed{u(t) = \left(2\cos 3t + \frac{4}{3}\sin 3t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left(\cos 2t + \frac{1}{2}\sin 2t\right)\begin{bmatrix} 1 \\ 4 \end{bmatrix}}$$

3. (12+4+4) Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & 9 \end{bmatrix}$

(a) Find the LDL^T-decomposition of the matrix A. (L is lower triangular and D is a diagonal matrix.)

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & 9 \end{bmatrix} \xrightarrow[2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$U = DL^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

(b) What is the determinant of A?

$$\det A = \det L \cdot \det D \cdot \det L^T$$
$$= 1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = \boxed{6}$$

(c) Is A positive-definite? Explain.

$\boxed{\text{Yes, since its 3 pivots } (1, 3, 2) \text{ are all } > 0}$

4. (4+4+4+4) This problem has four unrelated parts.

(a) Given $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

Find the matrix which has v as an eigenvector.

$Av = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ ✗ $\boxed{Bv = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3v}$ ✓ $Cv = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ ✗

(b) If A has eigenvalues $-2, 1$ and eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively.

Find the eigenvalues, and eigenvectors of $A^3 + I$.

A^3+I has the same eigenvectors, and eigenvalues $(-2)^3+1=-7$ and $1^3+1=2$

(c) If $A = P \cdot D \cdot P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$, calculate A^{10}.

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -2^{10}+2 & -2^{10}+1 \\ 2^{11}-2 & 2^{11}-1 \end{bmatrix}}$$

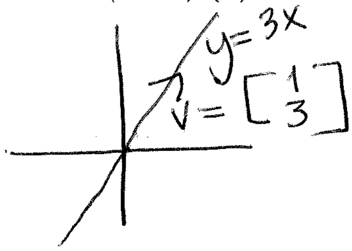
(d) If A is a symmetric, positive-definite matrix, which of the following are always positive-definite : $A - 2I$, $A^T A$, $A^2 - 2A + 2I$. Explain your answer.

$A-2I$: $\boxed{\text{No}}$, for example $I-2I$ is not pos. def.

A^TA : $\boxed{\text{Yes}}$ such matrices are always pos. def. if $\det A \neq 0$

$A^2-2A+2I$: $\boxed{\text{Yes}}$ it has eigenvalues $\lambda^2-2\lambda+2$ $=(\lambda-1)^2+1>0$.

5. (8+12) **(a)** Find the projection matrix P of any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ onto the line $y = 3x$.



$$A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} (10)^{-1} \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}}$$

(b) Find the equation of the plane $z = ax + by + c$ fitting best to the data: (1,1,0),(1,0,1),(-1,0,2),(1,-1,3), and calculate the error i.e. e^Te

$$\begin{aligned} a + b + c &= 0 \\ a + \phantom{b+{}} c &= 1 \\ -a + \phantom{b+{}} c &= 2 \\ a - b + c &= 3 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$A^TA \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 2 \\ 0 & 2 & 0 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 3 & 5 \end{array}\right]$$

$$c = 5/3 \qquad b = -3/2 \qquad a = -1/3$$

$$\boxed{z = \frac{-1}{3}x - \frac{3}{2}y + \frac{5}{3}}$$

$$e = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ -3/2 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4/3 \\ 2 \\ 17/6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1/3 \\ 0 \\ 1/6 \end{bmatrix}}$$

$$e^Te = 1^2 + \frac{1}{9} + \frac{1}{36} = \boxed{41/36}$$