M E T U
Northern Cyprus Campus

	Applied Mathematics for Engineers Final Exam	
Code : *Math 210* Acad.Year: *2010-2011* Semester : *Spring* Date : *07.6.2011* Time : *13:00* Duration : *180 minutes*	Last Name: Name : Department: Signature: 7 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	Student No: Section:

1	2	3	4	5	6	7		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!
Please do not write on your desk!

1. (3+3+4) Let $< f, g >= \int_{-1}^{1} x^2 f(x)g(x)dx$ for continuous functions f, g defined on $[-1, 1]$

(a) Show that x^m is orthogonal to x^n if m is even and n is odd.

$$\langle x^m, x^n \rangle = \int_{-1}^{1} x^2 \cdot x^m \cdot x^n \, dx = \int_{-1}^{1} x^{m+n+2} dx = \left. \frac{x^{m+n+3}}{m+n+3} \right|_{-1}^{1}$$

$$= \begin{cases} 0 & \text{if } m+n \text{ is odd} \\ \frac{2}{m+n+3} & \text{if } m+n \text{ is even} \end{cases}$$

(b) Find the norm of x^n.

$$\|x^n\| = \langle x^n, x^n \rangle^{1/2} = \left(\int_{-1}^{1} x^2 \cdot x^n \cdot x^n dx \right)^{1/2} = \left(\int_{-1}^{1} x^{2n+2} dx \right)^{1/2} \underset{x^{2n+2} \text{ is even}}{=} \left(2 \cdot \int_{0}^{1} x^{2n+2} dx \right)^{1/2}$$

$$= \left(2 \cdot \left. \frac{x^{2n+3}}{2n+3} \right|_0^1 \right)^{1/2} = \sqrt{\frac{2}{2n+3}}.$$

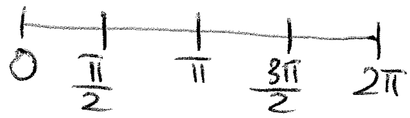
(c) Compute $\mathbf{Proj}_{x^n} x^m$.

$$\text{Proj}_{x^n} x^m = \frac{\langle x^m, x^n \rangle}{\langle x^n, x^n \rangle} x^n = \left(\frac{\frac{1^{m+n+3} - (-1)^{m+n+3}}{m+n+3}}{\frac{2}{2n+3}} \right) x^n$$

2. (4+4+4+4+4)Given $f(x) = \sin(x)$ and $g(x) = 3\sin(x) + 2\cos(x)$

(a) For $N = 4$, discretize $f(x)$ and $g(x)$ on $[0, 2\pi]$ to get $\vec{f}$ and $\vec{g}$.

For $N=4$, $h = \frac{2\pi}{4} = \frac{\pi}{2}$, sample points are $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



$$\vec{f} = \begin{bmatrix} f(0) \\ f(\pi/2) \\ f(\pi) \\ f(3\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g(0) \\ g(\pi/2) \\ g(\pi) \\ g(3\pi/2) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}$$

(b) Compute $\vec{f} \circledast \vec{g}$.

$N=4$ implies that cyclic convolution is done Mod 4.

$$\begin{array}{rrrrrrr}
 & & & 2 & 3 & -2 & -3 \\
\times & & & 0 & 1 & 0 & -1 \\
\hline
 & & & -2 & -3 & 2 & 3 \\
 & & 0 & 0 & 0 & 0 & \\
 & 2 & 3 & -2 & -3 & & \\
+0 & 0 & 0 & 0 & & & \\
\hline
0 & 2 & 3 & -4 & -6 & 2 & 3
\end{array}$$

$\vec{f} \circledast \vec{g} = (-6, 4, 6, -4)$

(c) Write F_4 and F_4^{-1}.

$\omega = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i \qquad \overline{\omega} = -i$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \qquad F_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

(d) Find **DFT** of $\overrightarrow{f}$ and $\overrightarrow{g}$.

$$\vec{c} = F_4^{-1} \cdot \vec{f} \qquad \vec{d} = F_4^{-1} \cdot \vec{g}$$

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ -2i \\ 0 \\ +2i \end{bmatrix}$$

$$\frac{1}{4} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 4-6i \\ 0 \\ 4+6i \end{bmatrix}$$

(e) Verify the following equation:

$$F_4^{-1}(\overrightarrow{f} \circledast \overrightarrow{g}) = 4(F_4^{-1}\overrightarrow{f} \cdot F_4^{-1}\overrightarrow{g})$$

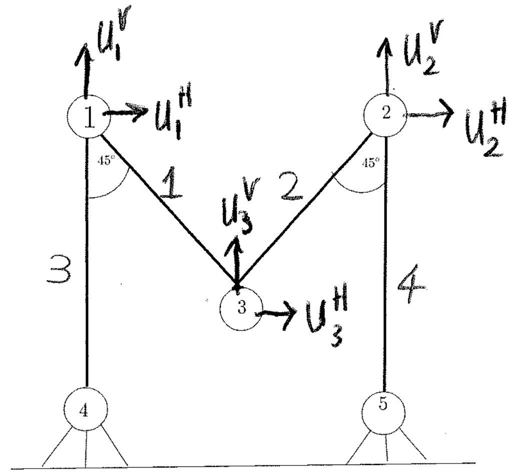
$$F_4^{-1}(\vec{f} \circledast \vec{g}) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 6 \\ -4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ -12-8i \\ 0 \\ -12+8i \end{bmatrix} = \begin{bmatrix} 0 \\ -3-2i \\ 0 \\ -3+2i \end{bmatrix} \checkmark$$

$$4 \cdot F_4^{-1}\vec{f} \cdot F_4^{-1}\vec{g} = 4 \cdot \left(\frac{1}{4} \begin{bmatrix} 0 \\ -2i \\ 0 \\ 2i \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 0 \\ 4-6i \\ 0 \\ 4+6i \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -3-2i \\ 0 \\ -3+2i \end{bmatrix}$$

3. (6+4+6+4) Given the following truss

(a) Find the elongation matrix A. i.e. $e = Au$.

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \end{bmatrix}$$



(b) Explain why the truss is unstable.

$$A \xrightarrow[\frac{2}{\sqrt{2}} R_2]{-\frac{2}{\sqrt{2}} R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Rank $A = 4$

$4 < 6$

So, unstable

(c) Find a basis for the null space of A.

Free variables are u_3^H and u_3^V

$u_1^H = u_1^V + u_3^H - u_3^V$

$u_1^V = 0$

$u_2^H = -u_2^V + u_3^H + u_3^V$

$u_2^V = 0$

$u_3^H = u_3^H$

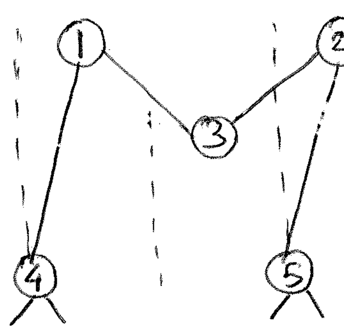
$u_3^V = u_3^V$

So
$$\begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_3^H + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_3^V$$

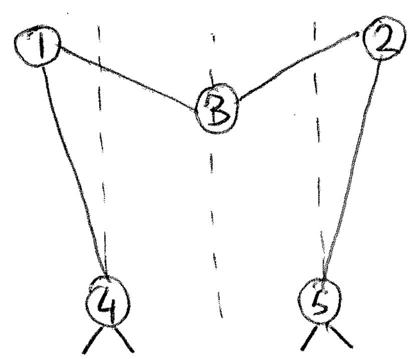
basis

(d) Draw the mechanism(s) corresponding to the basis element(s) of the null space of A.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow$$

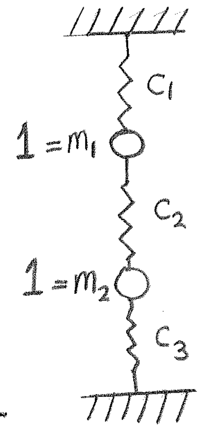


$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow$$



Full Name:________________________________

4. (3+4+3+5) Consider the spring-mass system with two masses $m_1 = 1, m_2 = 1$ and three springs with spring constants c_1, c_2, c_3. Both ends are fixed. Assume that there is no friction and external force acting, and the masses are moving up and down. Let $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ be the displacements of the masses at any time t



(a) Write the elongations e_1, e_2, e_3 of the springs in terms of u_1 and u_2.

$$e_1 = u_1 \qquad e_2 = u_2 - u_1 \qquad e_3 = -u_2$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Find the stiffness matrix K.

$$K = A^T \cdot C \cdot A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

(c) Find $M^{-1}K$, and calculate $Trace(M^{-1}K)$ and $det(M^{-1}K)$.

$$M^{-1} \cdot K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$\text{Trace}(M^{-1}K) = c_1 + 2c_2 + c_3$$

$$\det(M^{-1}K) = c_1 \cdot c_2 + c_1 \cdot c_3 + c$$

(d) Find the relation between the spring constants so that one of the natural frequencies is twice the other one. (**Do Not Solve** for c_1, c_2, c_3.)

$$\det(M^{-1}K - \lambda I) = \lambda^2 - (c_1 + 2c_2 + c_3)\lambda + c_1 \cdot c_2 + c_1 \cdot c_3 + c_2 \cdot c_3$$

$$\lambda_1 + \lambda_2 = \text{Trace}(M^{-1}K) = c_1 + 2c_2 + c_3 \qquad \omega_1 = \sqrt{\lambda_1}$$

$$\lambda_1 \cdot \lambda_2 = \det(M^{-1}K) = c_1 \cdot c_2 + c_1 \cdot c_3 + c_2 \cdot c_3 \qquad \omega_2 = \sqrt{\lambda_2}$$

If $\omega_1 = 2\omega_2$, then $\lambda_1 = 4\lambda_2$, hence $\lambda_1 + \lambda_2 = 5\lambda_2$

$$\lambda_1 \cdot \lambda_2 = 4\lambda_2^2$$

We get

$$\boxed{4\left(\frac{c_1 + 2c_2 + c_3}{5}\right)^2 = c_1 \cdot c_2 + c_1 \cdot c_3 + c_2 \cdot c_3}$$

5. (3+3+3+3+3) This problem has five unrelated parts.

(a) Find the LU-decomposition of $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

(b) Prove or disprove: If A and B are symmetric, then so is $A \cdot B$.

We disprove it!

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \leftarrow \text{Not symmetric.}$$

(c) Find the value(s) of a so that $C = \begin{bmatrix} a & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{bmatrix}$ is positive definite.

Upper Determinants: must be positive

1) $a > 0$

2) $2a - 1 > 0 \Rightarrow a > \frac{1}{2}$

3) $2a^2 - 2 > 0 \Rightarrow a < -1$ or $a > 1$

Hence, $\boxed{a > 1}$

(d) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$\det(A - \lambda I) = \left| \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right| = (3-\lambda)^3, \text{ hence } \underline{\lambda = 3} \text{ only.}$$

$\lambda = 3$

$$A - 3I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ we get } \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_2$$

(e) For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find P(invertible) and D(diagonal) matrices such that $P \cdot A = D \cdot P$

$P \cdot A = D \cdot P \Rightarrow A = P^{-1} \cdot D \cdot P$, we need to do diagonalization

$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda-3)\cdot(\lambda+1)$, eigenvalues are 3&

$\boxed{\lambda = 3}$

$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$, Hence $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will be a basis for eigenspace of

$\boxed{\lambda = -1}$

$A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, hence $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will be a basis for eigenspace of

Then, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

6. (10) Find the equation of the line $y = a + bx$ fitting best to the data: (-2,-13),(-1,-9),(1,-1),(4,11).

We get the following system!

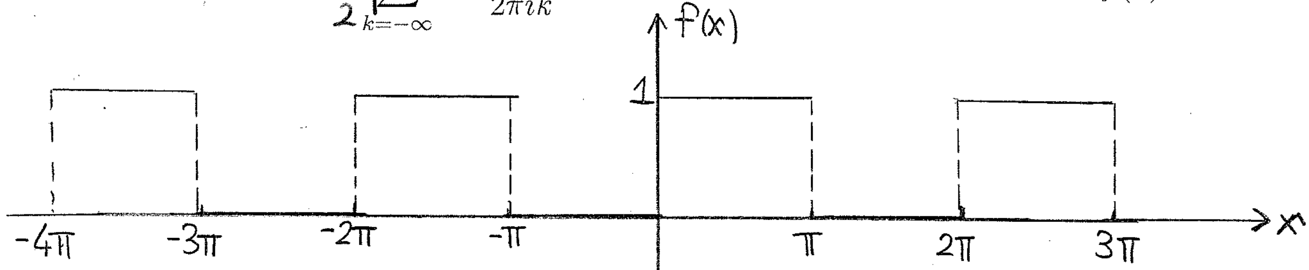
$$\underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}}_{= A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} -13 \\ -9 \\ -1 \\ 11 \end{bmatrix}}_{= b}$$

By using the method of Least Squares we need to solve $A^T \cdot A \cdot \vec{u} = A^T \cdot b$ to find the best fitting line.

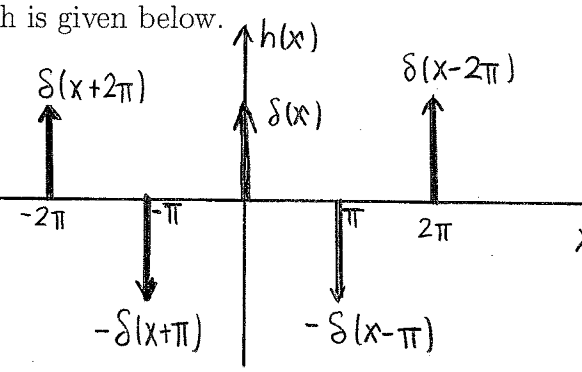
$$A^T \cdot A \cdot \vec{u} = A^T b \Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 22 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -12 \\ 78 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{84} \cdot \begin{bmatrix} 22 & -2 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ 78 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} \qquad \boxed{y = -5 + 4x}$$

7. (4+6) Given $\frac{1}{2} + \sum_{k=-\infty}^{\infty} \frac{(1-(-1)^k)}{2\pi i k} e^{ikx}$ as the Fourier series of function $f(x)$ below.



(a) Find the Fourier Series of the function $h(x)$ whose graph is given below.

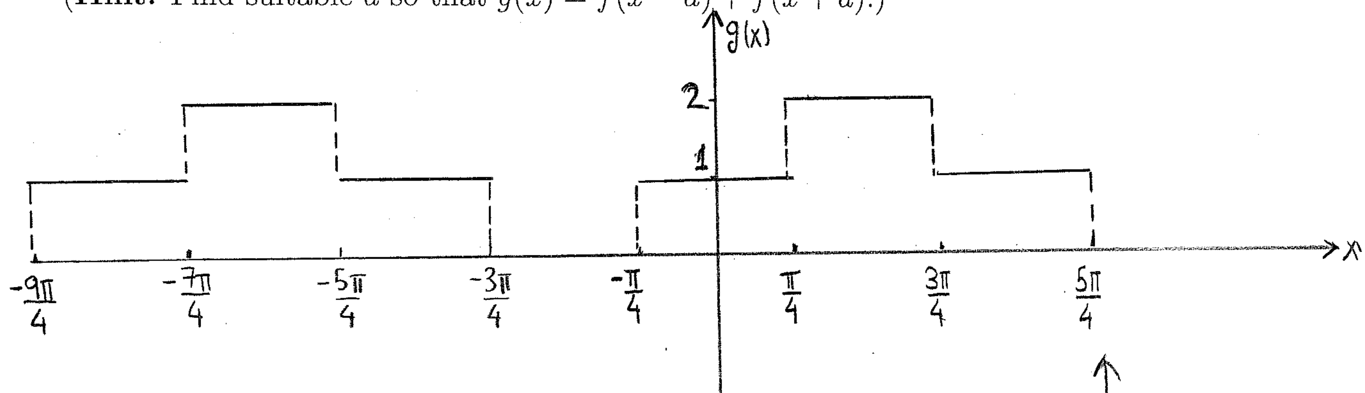


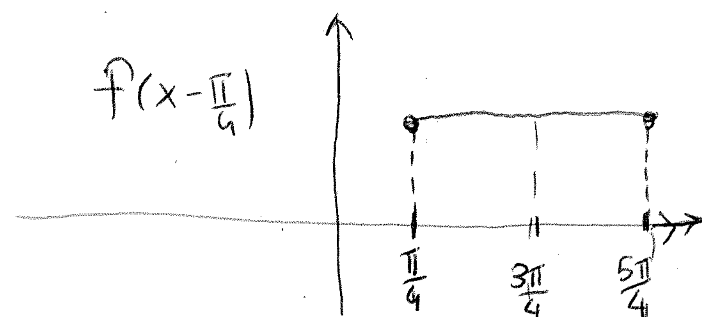
$h(x) = f'(x)$

Hence, $h(x) = 0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(1-(-1)^k)}{2\pi \cdot i \cdot k} \cdot i \cdot k \cdot e^{ikx}$

$= \sum_{k=-\infty}^{\infty} \frac{(1-(-1)^k)}{2\pi} e^{ikx}$

(b) Given $g(x)$ below, find the Fourier series of $g(x)$.
(**Hint:** Find suitable a so that $g(x) = f(x-a) + f(x+a)$.)

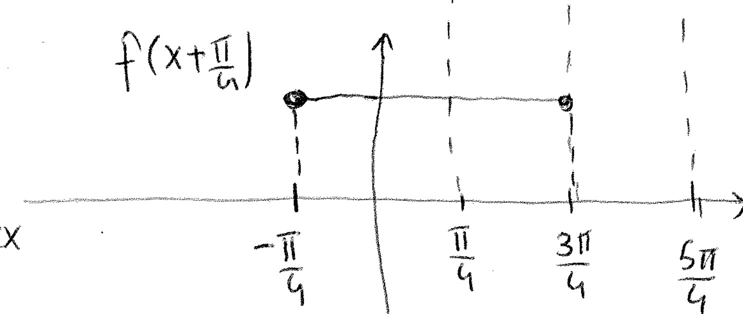




$g(x) = f\left(x - \frac{\pi}{4}\right) + f\left(x + \frac{\pi}{4}\right)$

By shifting properties of Fourier Transform,

$g(x) = \frac{1}{2} + \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} e^{-i k \cdot \frac{\pi}{4}} \cdot \left(\frac{1-(-1)^k}{2\pi i k}\right) e^{ikx}$



$+ \frac{1}{2} + \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} e^{-i k \cdot (-\frac{\pi}{4})} \left(\frac{1-(-1)^k}{2\pi i k}\right) e^{ikx}$