METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 2	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *FALL* Date : *07.12.2013* Time : *13:40* Duration : *110 min*	Last Name: Name : SOLUTIONS Student # : Signature : 5 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (16)	2. (20)	3. (16)	4. (24)	5. (24)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(2×8=16pts)* This problem has two (unrelated) parts.

(A) Find the critical points of $f(x,y) = x^2 + y^2 + x^2y - 2y - 12$.

$\nabla f = \langle 2x + 2xy,\ 2y + x^2 - 2\rangle$

$\nabla f = \langle 0,0\rangle \rightsquigarrow 2x + 2xy = 0 \rightsquigarrow 2x(1+y) = 0 \nearrow x=0$ or $\searrow y=-1$

and

$2y + x^2 - 2 = 0$ ← plug into other equation.

if $x=0$ then $2y - 2 = 0$, $y = 1$ $\boxed{(0,1)}$

if $y=-1$ then $-2 + x^2 - 2 = 0$, $x^2 = 4$, $x = \pm 2$ $\boxed{(\pm 2, -1)}$

(B) For what values of a does $f(x,y) = 2x^2 + 3y^2 + axy$ have a maximum or minimum at $(0,0)$? Is it a max or min?

(f is continuous & differentiable on $\mathbb{R}^2$ so any global max or min must be at a point with $\nabla f = \langle 0,0\rangle$)

$\nabla f = \langle 4x + ay,\ 6y + ax\rangle \rightsquigarrow \nabla f(0,0) = \langle 0,0\rangle$ so $(0,0)$ is a critical pt.

→ when is it max/min?

2nd Derivative Test.

$f_{xx} = 4$, $f_{yy} = 6$, $f_{xy} = a$ } Max or min if $D = 4\cdot 6 - a^2 > 0$

$24 > a^2$

$2\sqrt{6} > |a|$

$\boxed{-2\sqrt{6} < a < 2\sqrt{6}}$

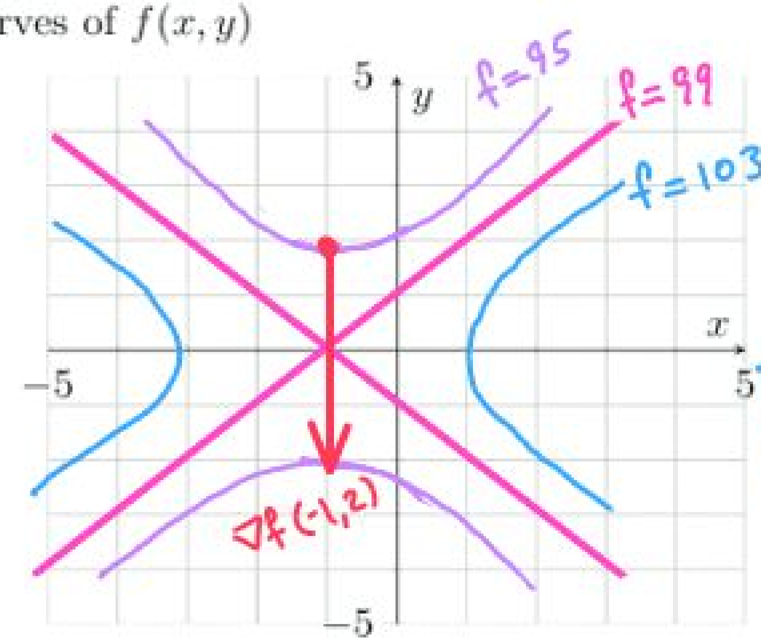
$\left.\begin{matrix} f_{xx} = 4 \\ f_{yy} = 6 \end{matrix}\right\} > 0$

so any critical point must be a $\boxed{\text{Minimum}}$

2. *(2+6+6+6=20pts)* Suppose that the height function of a certain mountain is given by

$$f(x, y) = x^2 + 2x - y^2 + 100.$$

(A) Sketch several level curves of $f(x, y)$



Note:

$f(x,y) = (x+1)^2 - y^2 + 99$

So $z = f(x,y)$ is a hyperbolic paraboloid (saddle) centered at $(-1, 0, 99)$

(B) Find the gradient of f at the point $(-1, 2)$ and place the resulting vector on the graph in part (a).

$$\nabla f = \langle 2x+2, -2y \rangle$$
$$\nabla f(-1,2) = \langle -2+2, -4 \rangle$$
$$= \boxed{\langle 0, -4 \rangle}$$

(C) Find the rate of change of the height function at the instant when one starts to move from the point $(-1, 2)$ towards the origin. Is the height function increasing or decreasing at that instant?

This rate of change is given by the directional derivative in the direction of $\underline{v} = -\langle -1, 2 \rangle = \langle 1, -2 \rangle$

$$\underline{u} = \frac{\langle 1,-2 \rangle}{|\langle 1,-2 \rangle|} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$$

$$D_{\underline{u}} f(-1,2) = \nabla f(-1,2) \cdot \underline{u} = \langle 0, -4 \rangle \cdot \frac{1}{\sqrt{5}} \langle 1, -2 \rangle = \boxed{8/\sqrt{5}}$$

Height is $\boxed{\text{increasing}}$ because $8/\sqrt{5} > 0$.

(D) Are there any points where all directional derivatives are 0? What else can you say about these points (what makes them special)?

$0 = D_{\underline{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \underline{u}$ for all $\underline{u}$

$\Rightarrow 0 = D_{\langle 1,0 \rangle} f(x_0, y_0) = f_x(x_0, y_0)$

$\Rightarrow 0 = D_{\langle 0,1 \rangle} f(x_0, y_0) = f_y(x_0, y_0)$

$\rightarrow$ so $\nabla f(x_0, y_0) = \langle 0, 0 \rangle$ ((x_0, y_0) is a critical point)

$0 = 2x + 2 \rightsquigarrow x = -1$
$0 = -2y \rightsquigarrow y = 0$

$\boxed{(-1, 0)}$ is the only point with all directional derivatives $= 0$

It is a $\boxed{\text{critical point}}$. $D = 2(-2) - 0 < 0$ so it is a $\boxed{\text{saddle}}$

3. *(16pts)* Find the maximum and minimum values of the function $f(x,y,z) = x^2 - y^2$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 1$.

Use Lagrange multipliers:

$$\begin{cases} f = x^2 - y^2 \\ g = x^2 + 2y^2 + 3z^2 \end{cases}$$

$$\nabla f = \nabla g \cdot \lambda$$

$(\partial/\partial x)$ $\quad 2x = 2x \cdot \lambda$

$(\partial/\partial y)$ $\quad -2y = 4y \cdot \lambda$

$(\partial/\partial z)$ $\quad 0 = 6z \cdot \lambda \rightsquigarrow z = 0$ or $\lambda = 0$

and

$x^2 + 2y^2 + 3z^2 = 1$

If $\lambda = 0$ then $2x = 2x \cdot \lambda \rightsquigarrow 2x = 0$, $\boxed{x = 0}$

$-2y = 4y \cdot \lambda \rightsquigarrow -2y = 0$, $\boxed{y = 0}$

$x^2 + 2y^2 + 3z^2 = 1$

$3z^2 = 1$

$\boxed{z = \pm 1/\sqrt{3}}$

If $z = 0$ then $2x = 2x \cdot \lambda \rightsquigarrow x = 0$ or $\lambda = 1$

$-2y = 4y \cdot \lambda$

and

$x^2 + 2y^2 = 1$

If $\boxed{x = 0}$ then $x^2 + 2y^2 = 1$

$2y^2 = 1$

$\boxed{y = \pm 1/\sqrt{2}}$

If $\boxed{\lambda = 1}$ then $-2y = 4y \cdot \lambda$

$-2y = 4y$

$0 = 6y$

$\boxed{0 = y}$

and $x^2 + 2y^2 = 1$

$x^2 = 1$

$\boxed{x = \pm 1}$

We found the following points:

$$\boxed{(\pm 1, 0, 0) \qquad (0, \pm 1/\sqrt{2}, 0) \qquad (0, 0, \pm 1/\sqrt{3})}$$

Since $x^2 + 2y^2 + 3z^2 = 1$ is a closed and bounded set and $f = x^2 - y^2$ is continuous, by the Extreme Value Theorem there must be a global max & min. Plug in to find them.

$\boxed{f(\pm 1, 0, 0) = 1}$ Global Max

$\boxed{f(0, \pm 1/\sqrt{2}, 0) = -1/2}$ Global Min

$f(0, 0, \pm 1/\sqrt{3}) = 0$

Not Special :(

4. *(4×6=24pts)* Compute the following double integrals.

(A) $\int_0^1 \int_0^2 xy^2 + x \, dx\, dy$ $= \int_0^1 \int_0^2 x(y^2+1)\, dx\, dy$

$= \int_0^1 \frac{1}{2} x^2 (y^2+1) \Big|_{x=0}^{x=2} dy$

$= \int_0^1 2(y^2+1)\, dy$

$= 2\left(\frac{1}{3}y^3 + y\right) \Big|_{y=0}^{y=1} = 2\left(\frac{1}{3}+1\right) = \boxed{\frac{8}{3}}$

(B) $\int_0^1 \int_0^2 x^5 e^{x^3 y}\, dy\, dx$ $= \int_0^1 x^5 \frac{1}{x^3} e^{x^3 y} \Big|_{y=0}^{y=2} dx$

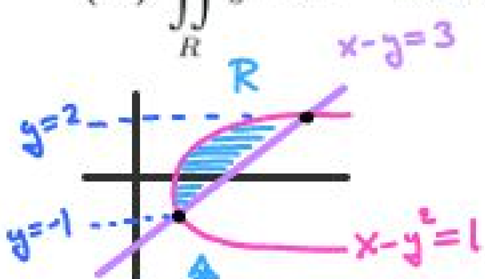
$= \int_0^1 x^2 e^{2x^3} - x^2 \, dx$

$= \frac{e^{2x^3}}{6} - \frac{1}{3}x^3 \Big|_{x=0}^{x=1}$

$= \frac{e^2}{6} - \frac{1}{3} - \frac{1}{6} = \boxed{\frac{e^2}{6} - \frac{1}{2}}$

(C) $\iint_R y^2 \, dA$ where R is the region enclosed by $x - y^2 = 1$ and $x - y = 3$.

$x = 1 + y^2$ and $x = 3 + y$



Intersection Points: $1 + y^2 = x = 3 + y$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \qquad y = -1, 2$

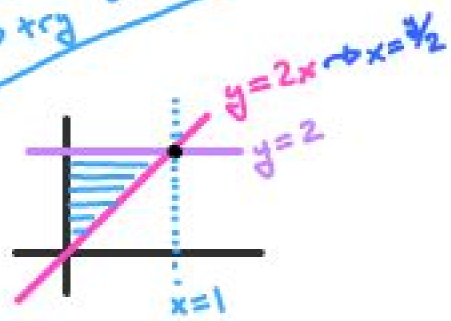
This region is best done dx dy

$\iint_R y^2\, dA = \int_{y=-1}^{y=2} \int_{x=1+y^2}^{x=3+y} y^2\, dx\, dy = \int_{y=-1}^{y=2} xy^2 \Big|_{x=1+y^2}^{x=3+y} dy = \int_{y=-1}^{y=2} y^2\big((3+y)-(1+y^2)\big)\, dy$

$= \frac{2}{3}y^3 + \frac{1}{4}y^4 - \frac{1}{5}y^5 \Big|_{y=-1}^{y=2} = \boxed{\frac{18}{3} + \frac{15}{4} - \frac{33}{5}}$

(D) $\iint_R \cos(y^2)\, dA$ where R is the region enclosed by $x = 0$, $y = 2$, and $y = 2x$.

We cannot integrate this function dy dx → try dx dy



$\iint_R \cos(y^2)\, dA = \int_{y=0}^{y=2} \int_{x=0}^{x=y/2} \cos(y^2)\, dx\, dy$

$= \int_{y=0}^{y=2} x \cos(y^2) \Big|_{x=0}^{x=y/2} dy$

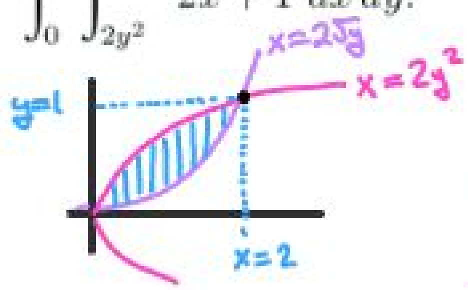
$= \int_{y=0}^{y=2} \frac{1}{2} y \cos(y^2)\, dy$

$= \frac{1}{4} \sin(y^2) \Big|_{y=0}^{y=2} = \boxed{\frac{1}{4}\sin(4)}$

5. *(4×6=24pts)* Transform the following integrals as indicated.

(A) Reverse the order of integration.

$$\int_0^1 \int_{2y^2}^{2\sqrt{y}} 2x + 1 \, dx \, dy.$$



Old Boundary ⟶ New Boundary

intersections: $x = 2\sqrt{y}$, $x = 2y^2$ ⟶ $y = 0$, $\cancel{y = 1}$

$x = 2\sqrt{y}$ ⟶ $y = (x/2)^2$

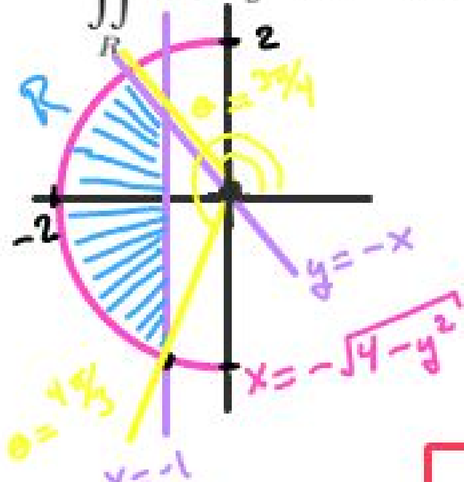
$x = 2y^2$ ⟶ $y = \sqrt{x/2}$

intersections: $y = \sqrt{x/2}$, $y = (x/2)^2$ ⟶ $x = 0$, $x = 2$

$$\boxed{\int_{x=0}^{x=2} \int_{y=(x/2)^2}^{y=\sqrt{x/2}} 2x+1 \, dy \, dx}$$

(B) Change to polar coordinates.

$$\iint_R x + 2y^2 \, dA \quad \text{where } R \text{ is the region with } x \geqslant -\sqrt{4-y^2},\ y \leqslant -x, \text{ and } x \leqslant -1.$$



Boundary

$x = -\sqrt{4-y^2}$

$x^2+y^2=4$

$\boxed{r = 2}$

$y = -x$

$y/x = -1$

$\theta = -\pi/4$

$\boxed{\theta = 3\pi/4}$

$x = -1$

$r\cos\theta = -1$

$r = -1/\cos\theta$

$\boxed{r = -\sec\theta}$

Intersection of $x=-1$ and $x=-\sqrt{4-y^2}$ is at $(-1, \pm\sqrt{3})$

$\boxed{\theta = 4\pi/3}$

$$\boxed{\int_{\theta=3\pi/4}^{\theta=4\pi/3} \int_{r=-\sec\theta}^{r=2} \left(r\cos\theta + 2(r\sin\theta)^2\right) r \, dr \, d\theta}$$

(Problem 5 continues here...)

(C) Substitute $u = \frac{x}{y}$, $v = xy$.

$\iint_R x^2 + y^2 \, dA$ where R is inside $xy = 1$, $xy = 2$, $x = y$, and $x = 3y$ for $x > 0$.

Boundary:

$xy = 1$	$xy = 2$	$x = y$	$x = 3y$
$\boxed{v = 1}$	$\boxed{v = 2}$	$x/y = 1$	$x/y = 3$
		$\boxed{u = 1}$	$\boxed{u = 3}$

Function: $\left.\begin{array}{l} u = x/y \\ v = xy \end{array}\right\} \to uv = \frac{x}{y} \cdot xy = x^2$ So $\boxed{x^2 + y^2 = uv + v/u}$

$v/u = xy / \cancel{x}/y = y^2$

Jacobian: $\left.\begin{array}{l} u = x/y \\ v = xy \end{array}\right\} \to du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$ So $\boxed{dx\,dy = \left|\frac{1}{2u}\right| du\,dv}$

$= |u_x v_y - u_y v_x| \, dx\,dy$

$= |1/y \cdot x - (-x/y^2) y| \, dx\,dy = 2 \left(\frac{x}{y}\right) dx\,dy$ u

$$\boxed{\int_{v=1}^{v=2} \int_{u=1}^{u=3} \left(uv + v/u\right) \frac{1}{2u} \, du\,dv}$$

(D) Substitute $x = \frac{u}{v}$, $y = \sqrt{v}$. (Careful This one is tricky.)

$\iint_R xy^2 + 1 \, dA$ where R is inside $x = y^2$, $x = 2y^2$, $xy^2 = 2$, and $xy^2 = 3$.

Boundary:

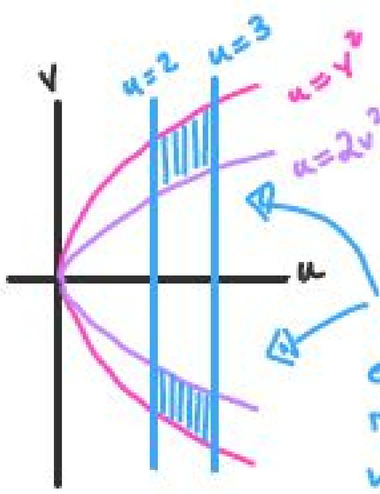
$x = y^2$	$x = 2y^2$	$xy^2 = 2$	$xy^2 = 3$
$u/v = v$	$u/v = 2v$	$u/v \, v = 2$	$u/v \, v = 3$
$\boxed{u = v^2}$	$\boxed{u = 2v^2}$	$\boxed{u = 2}$	$\boxed{u = 3}$

Function: $xy^2 + 1 = u/v \cdot v + 1 = \boxed{u + 1}$

Jacobian: $\left.\begin{array}{l} x = u/v \\ y = \sqrt{v} \end{array}\right\} dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$ $\boxed{dx\,dy = \frac{1}{2} v^{-3/2} \, du\,dv}$

$= |x_u y_v - x_v y_u| \, du\,dv$

$= \left| 1/v \left(\frac{1}{2\sqrt{v}}\right) - 0 \right| du\,dv$



We can integrate over either of these two regions... but notice that we <u>must</u> integrate dv <u>first</u>

$\left\{\begin{array}{l} u = v^2 \rightsquigarrow v = \sqrt{u} \\ u = 2v^2 \rightsquigarrow v = \sqrt{u/2} \end{array}\right.$

$$\boxed{\int_{u=2}^{u=3} \int_{v=\sqrt{u/2}}^{v=\sqrt{u}} (u+1) \cdot \frac{1}{2} v^{-3/2} \, dv\,du}$$