METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 2	
Code : *MAT 120* Acad.Year: *2011-2012* Semester : *Spring* Date : *28.4.2012* Time : *9:40* Duration : *??? minutes*	Last Name: Name : Student No.: Department: Section: Signature:
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (12)	2 (16)	3 (6)	4 (12)	5 (12)	6 (18)	7 (24)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

Good luck!

1. *(6+6 pts)* Let $f(x,y) = x^2y + \frac{1}{6}y^3 - y + 1$.

(a) Find the critical points of f and classify them as local maxima, local minima, or saddles.

$$\left.\begin{aligned} f_x &= 2xy = 0 \\ f_y &= x^2 + \frac{y^2}{2} - 1 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x=0 &\Rightarrow y^2 = 2 \Rightarrow y = \mp\sqrt{2} \\ y = 0 &\Rightarrow x^2 = 1 \Rightarrow x = \mp 1 \end{aligned}\right\} \Rightarrow$$

Points are: $(0,\sqrt{2})$, $(0,-\sqrt{2})$, $(1,0)$, $(-1,0)$

To classify: $f_{xx} = 2y$; $f_{yy} = y$; $f_{xy} = 2x$; $D(x,y) = 2y^2 - 4x^2$

$D(0,\sqrt{2}) = 4 > 0$; $f_{xx}(0,\sqrt{2}) = 2\sqrt{2} > 0 \Rightarrow f$ has local min at $(0,\sqrt{2})$

$D(0,-\sqrt{2}) = 4 > 0$; $f_{xx}(0,-\sqrt{2}) = -2\sqrt{2} < 0 \Rightarrow f$ has local max at $(0,-\sqrt{2})$

$$\left.\begin{aligned} D(1,0) &= -4 < 0 \\ D(-1,0) &= -4 < 0 \end{aligned}\right\} \Rightarrow f \text{ has saddle point at } (1,0) \text{ and } (-1,0)$$

(b) Find the maximum and minimum values of f on the ellipse $x^2 + \frac{y^2}{4} = 1$.

Using Lagrange Multipliers: $\nabla f = \lambda \nabla g$

$$\langle 2xy, x^2 + \frac{y^2}{2} - 1 \rangle = \lambda \langle 2x, \frac{y}{2} \rangle$$

$$\Rightarrow \left.\begin{aligned} 2xy &= 2x\lambda \\ x^2 + \frac{y^2}{2} - 1 &= \lambda \frac{y}{2} \end{aligned}\right\} \Rightarrow$$

$2x(y-\lambda) = 0$

if $x = 0$ then $\frac{y^2}{4} = 1 \Rightarrow y = \mp 2$ and $\lambda = \mp 1$

if $y = \lambda$ then $x^2 + \frac{\lambda^2}{2} - 1 = \frac{\lambda^2}{2} \Rightarrow x = \mp 1$ and $\lambda = 0$

Points are: $(0,-2)$; $(0,2)$; $(1,0)$; $(-1,0)$

$$\left.\begin{aligned} f(0,-2) &= \frac{5}{3} \\ f(0,2) &= \frac{1}{3} \\ f(1,0) &= f(-1,0) = 1 \end{aligned}\right\}$$

f has absolute max at $(0,-2)$ and its value is $5/3$

f has absolute min at $(0,2)$ and its value is $1/3$

2. *(6+6+6pts)* Evaluate the following integrals:

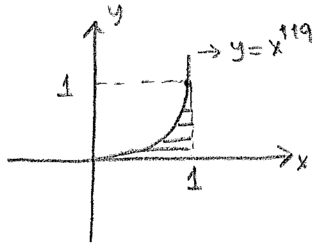
(a) $\int_1^3 \int_0^2 xy^2 + x \, dx \, dy = \int_1^3 \frac{x^2y^2}{2} + \frac{x^2}{2} \Big|_0^2 dy = \int_1^3 2y^2 + 2 \; dy$

$= \frac{2y^3}{3} + 2y \Big|_1^3$

$= 24 - \frac{8}{3}$

$= \frac{64}{3}$

(b) $\int_0^1 \int_{y^{1/119}}^1 \sin(x^{120}) dx\, dy = \int_0^1 \int_0^{x^{119}} \operatorname{Sin}(x^{120}) \, dy\, dx = \int_0^1 \operatorname{Sin}(x^{120}) \, y \Big|_0^{x^{119}} dx$

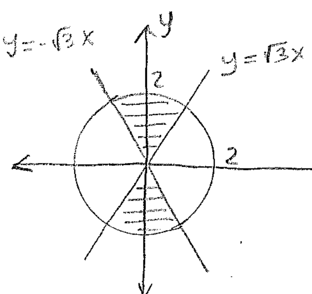


$= \int_0^1 \underbrace{\operatorname{Sin}(x^{120})}_{u} \underbrace{x^{119} dx}_{\frac{du}{120}}$

$= \frac{1}{120} \int_0^1 \sin u \, du = \frac{1}{120} \operatorname{Cos} u \Big|_0^1$

$= \frac{\operatorname{Cos}(1) - 1}{120}$

(c) $\iint_R e^{-\sqrt{x^2+y^2}} \, dA$; where $R = \{(x, y) \in \mathbb{R}^2 \colon |y| \geq \sqrt{3}|x| \text{ and } x^2 + y^2 \leq 4\}$



Using Polar Coordinates;

$\int_{\pi/3}^{2\pi/3} \int_0^2 e^{-r} r \, dr \, d\theta + \int_{4\pi/3}^{5\pi/3} \int_0^2 e^{-r} r \, dr \, d\theta$

$\left[e^{-\sqrt{x^2+y^2}} \text{ is even function with respect to } y \right]$ OR: $2 \int_{\pi/3}^{2\pi/3} \int_0^2 e^{-r} r \, dr \, d\theta = 2 \int_{\pi/3}^{2\pi/3} -\frac{e^{-r}(r+1)}{2} \Big|_0^2 d\theta$

$= \int_{\pi/3}^{2\pi/3} (1 - 3e^{-2}) \, d\theta$

$= (1 - 3e^{-2}) \theta \Big|_{\pi/2}^{2\pi/3}$

$= (1 - 3e^{-2}) \frac{\pi}{3}$

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3. *(6 pts)* Compute $\iiint_R e^{y^2/x^2}\,dV$; where $R = \{0 \le x \le 1,\ x^3 \le y \le x,\ 0 \le z \le xy\}$.

$$= \int_0^1 \int_{x^3}^{x} \int_0^{xy} e^{y^2/x^2}\,dz\,dy\,dx = \int_0^1 \int_{x^3}^{x} e^{y^2/x^2} z \Big|_0^{xy} dy\,dx = \int_0^1 \int_{x^3}^{x} e^{y^2/x^2}\,xy\,dy\,dx =$$

$$= \int_0^1 \frac{x^3}{2} e^{y^2/x^2} \Big|_{x^3}^{x} dx = \int_0^1 -\frac{x^3}{2}\left(e^{x^4} - e\right) dx = -\frac{e^{x^4}}{8} + e \cdot \frac{x^4}{8} \Big|_0^1$$

$$= \frac{1}{8}$$

4. *(6+6 pts)* Let $R = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le 1 - |x| - |y| \text{ and } x \ge 0\}$. Write a triple integral, **do not evaluate**, to compute the volume of R in

(a) $dx\,dy\,dz$ order.

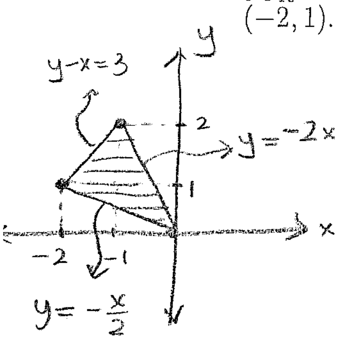
$$\int_0^1 \int_{z-1}^{0} \int_0^{1+y-z} 1\,dx\,dy\,dz + \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 1\,dx\,dy\,dz$$

(a) $dy\,dx\,dz$ order.

$$\int_0^1 \int_0^{1-z} \int_{x+z-1}^{1-x-z} 1\,dy\,dx\,dz$$

5. *(8+8 pts)* Evaluate the following integrals:

(a) $\iint_R 3x + 3y + e^{2x+y}\, dA$; where R is the triangle with vertices $(0,0)$, $(-1,2)$, and $(-2,1)$. (Hint: $s = x + 2y$ and $t = 2x + y$.)



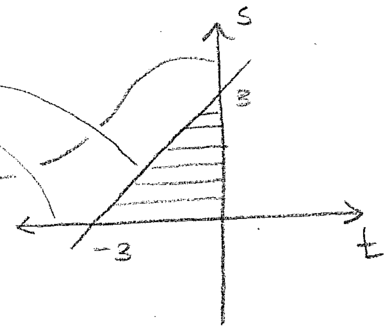
$s = x+2y$; $t = 2x+y$

$y - x = 3 \longrightarrow s - t = 3$

$y = -2x \longrightarrow t = 0$

$y = -\frac{x}{2} \longrightarrow s = 0$

New Region:



$x = \frac{2t-s}{3}$

$y = \frac{2s-t}{3}$

Jacobian:

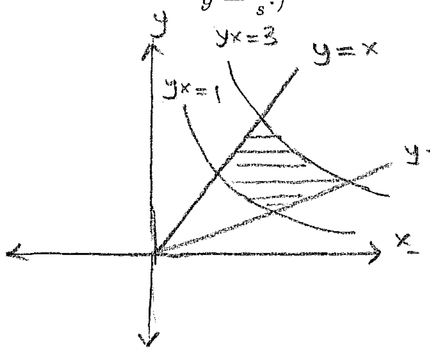
$J = \begin{vmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = \left|-\frac{3}{9}\right| = \frac{1}{3}$

Now, integral becomes; $\int_0^3 \int_{s-3}^0 (s+t+e^t)\frac{1}{3}\, dt\, ds = \frac{1}{3}\int_0^3 st + \frac{t^2}{2} + e^t \Big|_{s-3}^{0} ds$

$= \frac{1}{3}\int_0^3 1 - \left(s(s-3) + \frac{(s-3)^2}{2} + e^{s-3}\right) ds = \frac{1}{3}\left(-\frac{s^3}{2} + 3s^2 - \frac{7s}{2} - e^{s-3}\right)\Big|_0^3$

$= \frac{2 + e^{-3}}{3}$

(b) $\iint_R \frac{2x}{3y} e^{xy}\, dA$; where $R = \{1 \le xy \le 3 \text{ and } 0 \le y \le x \le 4y\}$. (Hint: $x = s$ and $y = \frac{t}{s}$.)



$s = x$

$t = xy$

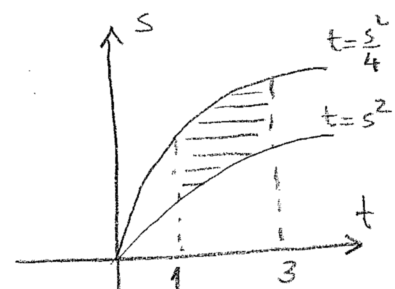
$y = x \longrightarrow t = s^2$

$y = \frac{x}{4} \longrightarrow t = \frac{s^2}{4}$

$yx = 1 \longrightarrow t = 1$

$yx = 3 \longrightarrow t = 3$

New Region:



$x = s$

$y = \frac{t}{s}$; $J = \begin{vmatrix} 1 & -\frac{t}{s^2} \\ 0 & \frac{1}{s} \end{vmatrix} = \frac{1}{s}$

Jacobian:

Now, integral becomes; $\int_1^3 \int_{\sqrt{t}}^{2\sqrt{t}} \frac{2}{3}\cdot\frac{s^{\cancel{2}}}{t} e^t \frac{1}{\cancel{s}}\, ds\, dt = \int_1^3 \frac{e^t}{3t} s^2 \Big|_{\sqrt{t}}^{2\sqrt{t}} dt$

$= \int_1^3 e^t\, dt = e^t \Big|_1^3$

$= e^3 - e$

6. *(6+6+6 pts)*

Compute the following line integrals:

(a) $\int_C 8x + 4\, ds$; where C is the curve along $y = x^2 + x$ from $(0,0)$ to $(1,2)$

Parametrization of C: $x = t$; $y = t^2 + t$; $0 \leq t \leq 1$.

Integral becomes; $\int_0^1 8t+4 \sqrt{1^2 + (2t+1)^2}\, dt = \int_0^1 \underbrace{(8t+4)}_{du} \sqrt{\underbrace{4t^2+4t+2}_{u}}\, dt$

$= \dfrac{(4t^2+4t+2)^{3/2}}{3/2} \Big|_0^1$

$= \frac{2}{3}\left(10^{3/2} - 2^{3/2}\right)$

(b) $\int_C xy^2\, dx + y\, dy$; where C is the curve along $x^2 - y^2 = 1$ from $(1,0)$ to $(\sqrt{2}, 1)$

Parametrization of C: $x = t$; $y = \sqrt{t^2-1}$; $1 \leq t \leq \sqrt{2}$

Integral becomes: $\int_1^{\sqrt{2}} t(t^2-1)\, dt + \cancel{\sqrt{t^2-1}} \cdot \left(\frac{1}{2} \frac{2t}{\cancel{\sqrt{t^2-1}}}\right) dt$

$= \int_1^{\sqrt{2}} t^3\, dt = \frac{t^4}{4} \Big|_1^{\sqrt{2}}$

$= 1 - \frac{1}{4} = \frac{3}{4}$

(c) $\int_C \mathbf{F} \cdot d\mathbf{r}$; where $\mathbf{F} = \langle \frac{1}{x+y^2}, \frac{2y}{x+y^2} + 1 \rangle$ and C is the curve parametrized by $\mathbf{r}(t) = \{(\cos t, \sin t) \text{ for } -\frac{\pi}{2} \leq t \leq 0\}$

$= \int_{-\frac{\pi}{2}}^{0} \left\langle \frac{1}{\cos t + \sin^2 t}, \frac{2\sin t}{\cos t + \sin^2 t} + 1 \right\rangle \cdot \langle -\sin t, \cos t \rangle\, dt$

$= \int_{-\frac{\pi}{2}}^{0} \frac{-\sin t}{\cos t + \sin^2 t} + \frac{2 \sin t \cos t}{\cos t + \sin^2 t} + \cos t\ dt = \int_{-\frac{\pi}{2}}^{0} \frac{\overbrace{2\sin t \cos t - \sin t}^{u'}}{\underbrace{\sin^2 t + \cos t}_{u}} + \cos t\ dt$

$= \left(\ln|\sin^2 t + \cos t| + \sin t\right)\Big|_{-\frac{\pi}{2}}^{0}$

$= \ln 1 - (\ln 1 - 1) = 1$

7. *(8+8+8 pts)* Let $\mathbf{F}(x, y, z) = \langle Axy + Bx^2\sin^2 z, x^2 + ze^{yz}, ye^{yz} + x^3 \sin(2z)\rangle$

(a) Determine A and B so that $\mathbf{F}(x, y, z)$ is conservative.

$$\frac{\partial (Axy + Bx^2\sin^2 z)}{\partial y} = \frac{\partial (x^2 + ze^{yz})}{\partial x} \Rightarrow Ax = 2x \Rightarrow A = 2$$

$$\frac{\partial (Axy + Bx^2\sin^2 z)}{\partial z} = \frac{\partial (ye^{yz} + x^3\sin(2z))}{\partial x} \Rightarrow B2x^2\sin z \cos z = 3x^2 \sin 2z \Rightarrow B = 3$$

Just to check

$$\frac{\partial (x^2 + ze^{yz})}{\partial z} = \frac{\partial (ye^{yz} + x^3\sin(2z))}{\partial y} \checkmark$$

(b) For A and B found in part (a) evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$; where C_1 is the curve parametrized by $x(t) = \sin^{120}(t\pi)$ $y(t) = \cos(120t\pi)$ and $z(t) = \cos(t^{2012}\pi)$ for $t \in [0, 2]$.

$$\left.\begin{array}{l} x(0) = x(2) = 0 \\ y(0) = y(2) = 1 \\ z(0) = z(2) = 1 \end{array}\right\}$$

Since the curve is closed and F is conservative, our integral must be 0. (By Fundamental Theorem of Line Integrals)

(c) Let C_2 be the curve connecting $(5, 0, 0)$ to $(-5, 0, 0)$ obtained by intersecting $z = 1 - (\frac{x^2}{25} + y^2)$ and $z = x^2$. Compute $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

F is conservative means there is a potential function f such that;

$$\frac{\partial f}{\partial x} = 2xy + 3x^2\sin^2 z \Rightarrow f(x,y,z) = x^2y + x^3\sin^2(z) + H(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + ze^{yz} \Rightarrow \cancel{x^2} + \frac{\partial H}{\partial y} = \cancel{x^2} + ze^{yz} \Rightarrow H(y,z) = e^{yz} + G(z)$$

so, $f(x,y,z) = x^2y + x^3\sin^2 z + e^{yz} + G(z)$

$$\frac{\partial f}{\partial z} = ye^{yz} + x^3\sin(2z) \Rightarrow \cancel{ye^{yz}} + x^3 2\sin z\cos z + G'(z) = \cancel{ye^{yz}} + \cancel{x^3\sin(2z)}$$

$$\Rightarrow G(z) = C$$

So, $f(x,y,z) = x^2y + x^3\sin^2 z + e^{yz} + C$

Using F.T.L.I.;
$$\int_{C_2} F \cdot dr = f(-5,0,0) - f(5,0,0)$$
$$= (1 + C) - (1 + C) = 0$$