METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2011-2012* Semester : *Spring* Date : *24.3.2012* Time : *9:40* Duration : *110 minutes*	Last Name: Name : Student No.: Department: Section: Signature: 7 QUESTIONS ON 7 PAGES TOTAL 100 POINTS

1 (12)	2 (15)	3 (16)	4 (12)	5 (18)	6 (12)	7 (15)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed. **Good luck!**

1. *(2+2+4+4pts)* The following parts are about the line $\mathbf{r}(t) = \langle t+2, -1, 2t+3\rangle$ and the xz-plane.

(a) Write the equation of the xz-plane.

$0\cdot(x-0)+1\cdot(y-0)+0\cdot(z-0)=0$

(b) Write a normal vector to the xz-plane.

$\vec{n} = \langle 0,1,0\rangle$

(c) Show that the line $\mathbf{r}(t)$ and the xz-plane are parallel. (It may help to draw a picture.)

Direction vector of line $\vec{r}(t)$ is: $\langle 1,0,2\rangle$

Since $\langle 0,1,0\rangle \cdot \langle 1,0,2\rangle = 0$ (dot product of normal vector of our plane and direction vector of our line)

These vectors are perpendicular, which means the line $\vec{r}(t)$ is parallel to xz plane.

(d) Use dot products to find the distance between $\mathbf{r}(t)$ and the xz-plane. (You must show work.)

Let's pick one point from $\vec{r}(t)$, say $P:(2,-1,3)$. Also let's pick one point from xz plane, say $Q:(1,0,2)$.

Vector connecting Q to P; $\overrightarrow{QP} = \langle 1,-1,1\rangle$. Scalar projection of $\overrightarrow{QP}$ to $\vec{n}$ must be the distance between $\vec{r}(t)$ and xz plane

So, the distance $d = \left|\dfrac{\overrightarrow{QP}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\dfrac{\langle 1,-1,1\rangle\cdot\langle 0,1,0\rangle}{|\langle 0,1,0\rangle|}\right|$

$d = \left|\dfrac{1\cdot 0 - 1\cdot 1 + 1\cdot 0}{\sqrt{0^2+1^2+0^2}}\right| \rightarrow d = 1.$

2. *(5 × 3pts)* The following parts are about tangent planes and lines.

(a) Let $F(x, y, z) = 4x^2 + 4y^2 - z + 4$ be a function of three variables. Find the equation of the tangent plane of the level surface of $F = 4$ at the point (0,1,4).

Tangent Plane Eqn at $P:(x_0, y_0, z_0)$ to $F(x,y,z)$;

$$\frac{\partial F}{\partial x}(P)(x-x_0) + \frac{\partial F}{\partial y}(P)(y-y_0) + \frac{\partial F}{\partial z}(P)(z-z_0) = 0$$

$\frac{\partial F}{\partial x} = 8x$; $\frac{\partial F}{\partial y} = 8y$; $\frac{\partial F}{\partial z} = -1$

$\frac{\partial F}{\partial x}(0,1,4) = 0$; $\frac{\partial F}{\partial y}(0,1,4) = 8$; $\frac{\partial F}{\partial z}(0,1,4) = -1$

So, the tangent plane eqn : $0(x-0) + 8(y-1) - 1(z-4) = 0$

(b) Find the equation of the tangent plane of $z = x^2 + (y-2)^2 + 3$ at (0,1,4).

Tangent Plane Eqn at $P:(x_0, y_0)$ to $g(x,y) = x^2 + (y-2)^2 + 3$

$$z = g(P) + \frac{\partial g}{\partial x}(P)(x-x_0) + \frac{\partial g}{\partial y}(P)(y-y_0)$$

$g(0,1) = 4$; $\frac{\partial g}{\partial x} = 2x$; $\frac{\partial g}{\partial y} = 2(y-2)$

$\frac{\partial g}{\partial x}(0,1) = 0$; $\frac{\partial g}{\partial y}(0,1) = -2$

So the tangent plane eqn : $z = 4 + 0(x-0) - 2(y-1)$

OR $0(x-0) - 2(y-1) - (z-4) = 0$.

(c) Find the equation of the **line** tangent to the intersection of the surfaces of parts (a) and (b) at (0,1,4).

Basically this line is lying in both planes in part a) & b).

In other words, it is just the intersection of these planes.

Normal vector of plane in a) : $\vec{n}_1 = \langle 0, 8, -1 \rangle$

Normal vector of plane in b) : $\vec{n}_2 = \langle 0, -2, -1 \rangle$

So, the direction vector of the line $\vec{n}_1 \times \vec{n}_2 = \langle -10, 0, 0 \rangle$.

Also, we know that $(0,1,4)$ is on this line.

Line Eqn :
$x = 0 - 10t$
$y = 1 + 0t$
$z = 4 + 0t$

Name: ID:

3. *(4 × 4pts)* Determine whether or not the following limits exist. If they exist then find the limit. Explain your answer.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(xy)^2}{x^4+y^4}$.

Direction 1: x-axis $\displaystyle\lim_{(x,0)\to(0,0)} \frac{(x\cdot 0)^2}{x^4+0^4} = 0$

$\neq$ Limit D.N.E

Direction 2: $y=x$ line $\displaystyle\lim_{(x,x)\to(0,0)} \frac{(x\cdot x)^2}{x^4+x^4} = \frac{1}{2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x+y^2}$.

Direction 1: x-axis $\displaystyle\lim_{(x,0)\to(0,0)} \frac{e^0 \cdot \sin x}{x+0} = 1$

$\neq$ Limit DNE

Direction 2: y-axis $\displaystyle\lim_{(0,x)\to(0,0)} \frac{e^y \cdot \sin 0}{0+y^2} = 0$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^2y^3+y^4+x^8}$.

Direction 1: x-axis $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^4\cdot 0^2}{x^2\cdot 0^3+0^4+x^8} = 0$

$\neq$ Limit D.N.E.

Direction 2: $y=x^2$ $\displaystyle\lim_{(x,x^2)\to(0,0)} \frac{x^4\cdot(x^2)^2}{x^2(x^2)^3+(x^2)^4+x^8} = \frac{1}{3}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^3+y^3)\sin(x^2+y^2)}{(x+y)(x^2+y^2)}$.

$$\lim_{(x,y)\to(0,0)} \frac{(x^3+y^3)\sin(x^2+y^2)}{(x+y)(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x+y} \cdot \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$ if each exists.

$$= \lim_{(x,y)\to(0,0)} x^2-xy+y^2 \cdot \underbrace{\lim_{u\to 0} \frac{\sin u}{u}}_{1}$$ where $u=x^2+y^2$

$$= \lim_{(x,y)\to(0,0)} x^2-xy+y^2$$

$= 0$ since, x^2-xy+y^2 is continuous (polynomial) lim is just the value at $(0,0)$.

4. *(4× 3pts)* Let a, b, c be positive non-zero numbers and

$$\mathbf{r}(t) = \left\langle a\cos\left(\frac{t}{c}\right),\ a\sin\left(\frac{t}{c}\right),\ \frac{bt}{c}\right\rangle.$$

(a) Find a formula for b in terms of a and c so that $|\mathbf{r}'(t)| = 1$.

$$\vec{r}'(t) = \left\langle -\frac{a}{c}\sin\left(\frac{t}{c}\right), \frac{a}{c}\cos\left(\frac{t}{c}\right), \frac{b}{c}\right\rangle$$

$$\Rightarrow |\vec{r}'(t)| = \sqrt{\frac{a^2}{c^2}\sin^2\left(\frac{t}{c}\right) + \frac{a^2}{c^2}\cos^2\left(\frac{t}{c}\right) + \frac{b^2}{c^2}} = 1$$

$$\Rightarrow \sqrt{\frac{a^2+b^2}{c^2}} = 1 \quad \Rightarrow \quad a^2+b^2 = c^2$$

(b) Show that $|\mathbf{r}''(t)| = \dfrac{a}{c^2}$.

$$\vec{r}''(t) = \left\langle -\frac{a}{c^2}\cos\left(\frac{t}{c}\right), -\frac{a}{c^2}\sin\left(\frac{t}{c}\right), 0\right\rangle$$

$$\Rightarrow |\vec{r}''(t)| = \sqrt{\frac{a^2}{c^4}\cos^2\left(\frac{t}{c}\right) + \frac{a^2}{c^4}\sin^2\left(\frac{t}{c}\right) + 0^2}$$

$$\Rightarrow |\vec{r}''(t)| = \sqrt{\frac{a^2}{c^4}} = \frac{a}{c^2}$$

(c) Let $\mathbf{n}(t) = \dfrac{c^2}{a}\mathbf{r}''(t)$. Show that, for every real number t_0, the line through the point $\mathbf{r}(t_0)$ in the direction of $\mathbf{n}(t_0)$ both
(1) crosses the z-axis, and also
(2) is perpendicular to the z-axis.

Vector equation of defined line: $\vec{v}(t) = r(t_0) + t\,\vec{n}(t_0)$

Explicitly, $\vec{v}(t) = \left\langle a\cos\left(\frac{t_0}{c}\right) - t\cos\left(\frac{t_0}{c}\right), a\sin\left(\frac{t_0}{c}\right) - t\sin\left(\frac{t_0}{c}\right), \frac{bt_0}{c}\right\rangle$

$$\vec{v}(t) = \left\langle (a-t)\cos\left(\frac{t_0}{c}\right), (a-t)\sin\left(\frac{t_0}{c}\right), \frac{bt_0}{c}\right\rangle$$

When $t=a$, the point on the line is $\left(0, 0, \frac{bt_0}{c}\right)$ which is a point on z-axis. Therefore this line crosses the z-axis.

Direction vector for z-axis: $\langle 0,0,1\rangle$

Direction vector of our line: $\left\langle -\cos\left(\frac{t_0}{c}\right), -\sin\left(\frac{t_0}{c}\right), 0\right\rangle$ (coefficients of t in the eqn.)

$$\langle 0,0,1\rangle \cdot \left\langle -\cos\left(\frac{t_0}{c}\right), -\sin\left(\frac{t_0}{c}\right), 0\right\rangle = 0$$

This means, the line is perpendicular to z-axis.

5. *(3+3+6+6pts)* Suppose $f = f(x, y, z)$ with $x = x(p, r)$, $y = y(r)$ and $z = z(t)$; where $t = t(p, r)$. (All functions differentiable.) Given the following table of values:

$t(119, 120) = 1$	$t_p(119, 120) = \frac{1}{2}$	$t_r(119, 120) = 1$	
$x(119, 120) = 1$	$x_p(119, 120) = \frac{1}{2}$	$x_r(119, 120) = 2$	
$y(120) = 2$	$y_r(120) = 6$		
$z(1) = 0$	$z_t(1) = 2$		
$f(1, 2, 0) = 0$	$f_x(1, 2, 0) = 2$	$f_y(1, 2, 0) = 0$	$f_z(1, 2, 0) = 1$
$f_{px}(1, 2, 0) = 1$	$f_{py}(1, 2, 0) = 0$	$f_{pz}(1, 2, 0) = 1$	f_{pp} f_{rr}
$f_{rx}(1, 2, 0) = 2$	$f_{ry}(1, 2, 0) = 0$	$f_{rz}(1, 2, 0) = 2$	

(Remember that e.g. $f_x = \frac{\partial}{\partial x} f$ *and* $f_{rx} = \frac{\partial^2}{\partial r \partial x} f$*.)*

(a) Compute f_p at the point $(p, r) = (119, 120)$.

$$f_p = f_x \cdot x_p + f_y \overset{=0}{y_p} + f_z \cdot z_t \cdot t_p$$

when $p = 119$, $r = 120$; $x = 1$, $y = 2$, $t = 1$ and $z = 0$

So, at $p = 119$, $r = 120$: $f_p = 2 \cdot \frac{1}{2} + 0.0 + 1.2.\frac{1}{2} = 2$

(b) Compute f_r at the point $(p, r) = (119, 120)$.

$$f_r = f_x\, x_r + f_y \cdot y_r + f_z \cdot z_t \cdot t_r$$

when $p = 119$, $r = 120$; $x = 1$, $y = 2$, $t = 1$ and $z = 0$

So, at $p = 119$, $r = 120$: $f_r = 2.2 + 0.6 + 1.2.1 = 6$

(c) Use linear approximation to estimate f at $(p, r) = (120, 119)$.

Using f_p and f_r from part a) and b) we can write linear approximation (or tangent plane eqn at $p = 119$, $q = 120$ which is close to our point $p = 120$, $q = 119$)

$L(p, q) = f(1, 2, 0) + 2(p - 119) + 6(q - 120) = 2(p - 119) + 6(q - 120)$

Approximate value ~~of~~ f $= L(120, 119) = 2(120 - 119) + 6(119 - 120) = -4$.

(d) Let $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $\mathbf{v} = \langle \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \rangle$. Compute the mixed directional derivative $D^2_{\mathbf{uv}} f = D_{\mathbf{u}}(D_{\mathbf{v}} f)$ at the point $(p, r) = (119, 120)$.

BONUS

$$D_v f = \langle f_p, f_r \rangle \cdot \langle \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2} \rangle = \tfrac{\sqrt{2}}{2} f_p - \tfrac{\sqrt{2}}{2} f_r$$

$$D^2_{uv} f = D_u\left(\tfrac{\sqrt{2}}{2} f_p - \tfrac{\sqrt{2}}{2} f_r\right) = \langle \tfrac{\sqrt{2}}{2} f_{pp} - \tfrac{\sqrt{2}}{2} f_{rp}, \tfrac{\sqrt{2}}{2} f_{pr} - \tfrac{\sqrt{2}}{2} f_{rr} \rangle \cdot \langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2} \rangle$$

$$D^2_{uv} f = \frac{f_{pp} - f_{rp}}{2} + \frac{f_{pr} - f_{rr}}{2} = \frac{f_{pp} - f_{rr}}{2}$$

$$f_{pp} = (f_{xx} \cdot x_p + f_{xz} z_t \cdot t_p) x_p + f_x x_{pp} + (f_{zx} \cdot x_p + f_{zz} \cdot z_t \cdot t_p) . z_t \cdot t_p + f_z \cdot (z_{tt} t_p^2 + z_t \cdot t_{pp})$$

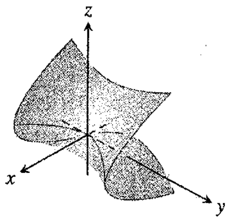
$$f_{rr} = (f_{xx} x_r + \cdots$$

6. *(3×4pts)* For the quadric surfaces pictured below,

(1) state their name (e.g. *"elliptic paraboloid"*),

(2) describe the x, y, and z traces (e.g. *"ellipse"*, *"hyperbola"*, *"line"*, *etc.*)

(3) and write an equation with coefficients ± 1 which would look like the surface (e.g. "$y^2 + z^2 + 1 = x^2$").

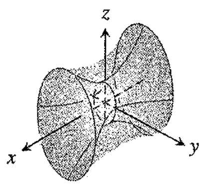


Name: Hyperpolic Paraboloid (Saddle)

Trace in $x = 1$: Hyperbola Trace in $y = 0$: Parabola

Trace in $z = 0$: Parabola

Equation: $x = y^2 - z^2$

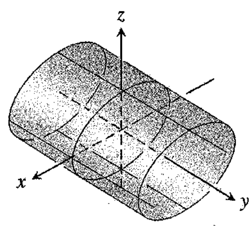


Name: Hyperboloid of One Sheet

Trace in $x = 0$: Ellipse (or Circle) Trace in $y = 0$: Hyperbola

Trace in $z = 0$: Hyperbola

Equation: $y^2 + z^2 - x^2 = 1$

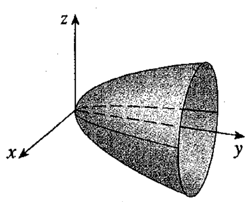


Name: Cylinder

Trace in $x = 0$: Two Lines Trace in $y = 0$: Ellipse (or Circle)

Trace in $z = 0$: Two Lines

Equation: $x^2 + z^2 = 1$



Name: Elliptic Parabola

Trace in $x = 0$: Parabola Trace in $y = 1$: Ellipse (or Circle)

Trace in $z = 0$: Parabola

Equation: $y = x^2 + z^2$

(The intersection of a surface by a plane is called the "trace" in the plane of the surface.)

7. *(9+6pts)* Compute the following.

(a) $f(x, y, z) = x^{y\cos(z)}$. Compute ∇f.

$$\vec{\nabla} f = \left\langle y\cos(z) \cdot x^{y\cos(z)-1},\ \ln\left(x^{\cos(z)}\right) \cdot x^{y\cos(z)},\ \ln\left(x^{y}\right) \cdot x^{y\cos z} \cdot -\sin z \right\rangle$$

(b) $x^2y + xyz + yz^2 + xy^2 = 4$. Compute $\dfrac{\partial z}{\partial x}$ at $(1, 2, -1)$.

Let $F(x,y,z) = x^2y + xyz + yz^2 + xy^2 - 4$ then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2xy + yz + y^2}{xy + 2yz}$$

So, $$\left.\frac{\partial z}{\partial x}\right|_{(1,2,-1)} = -\frac{2.1.2 + 2.-1 + 2^2}{1.2 + 2.2.-1} = -\frac{6}{-2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2,-1)} = 3.$$