METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 1	
Code : *MAT 119* Acad.Year: *2012-2013* Semester : *FALL* Date : *17.11.2012* Time : *9:40* Duration : *110 minutes*	Last Name: Name : Student No.: Department: Section: Signature: 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1 (16)	2 (20)	3 (8)	4 (12)	5 (14)	6 (18)	7 (5)	8 (7)		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(4×4pts)* Find the following limits, if they exist. Show your work.
Do not use L'Hospital's rule.

(A) $\lim\limits_{x\to 1} \dfrac{2x-2}{|x^3-x^2|} = \lim\limits_{x\to 1} \dfrac{2(x-1)}{x^2|x-1|}$

$\lim\limits_{x\to 1^+} \dfrac{2(x-1)}{x^2|x-1|} = \lim\limits_{x\to 1^+} \dfrac{2\cancel{(x-1)}}{x^2\cancel{(x-1)}} = 2$

$\neq$ No Limit.

$\lim\limits_{x\to 1^-} \dfrac{2(x-1)}{x^2|x-1|} = \lim\limits_{x\to 1^-} \dfrac{2\cancel{(x-1)}}{-x^2\cancel{(x-1)}} = -2$

(B) $\lim\limits_{x\to -1} \dfrac{x^2-1}{x^2-3x+2}$

$\lim\limits_{x\to -1} \dfrac{(x-1)(x+1)}{(x-1)(x-2)} = \dfrac{(-1-1)(-1+1)}{(-1-1)(-1-2)} = 0$

continuous at -1

(C) $\lim\limits_{x\to 0} \left[\dfrac{\cos x - 1}{\tan(2x)} + \dfrac{2}{x\csc x}\right]$ $\csc x = \dfrac{1}{\sin x}$, $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

$\lim\limits_{x\to 0} \underbrace{\dfrac{\cos x - 1}{2x}}_{\to 0} \underbrace{\dfrac{2x}{\tan(2x)}}_{\to 1} + \lim\limits_{x\to 0} 2\underbrace{\dfrac{\sin x}{x}}_{\to 1} = 0 + 2 = 2.$

(D) $\lim\limits_{x\to -\infty} \dfrac{3x-4}{\sqrt{4x^2+\pi x-3}}$

$\lim\limits_{x\to -\infty} \dfrac{x\left(3-\frac{4}{x}\right)}{\sqrt{4x^2\left(1+\frac{\pi}{4x}-\frac{3}{4x^2}\right)}} = \lim\limits_{x\to -\infty} \dfrac{x\left(3-\frac{4}{x}\right)}{|2x|\sqrt{1+\frac{\pi}{4x}-\frac{3}{4x^2}}}$

$= \lim\limits_{x\to -\infty} \dfrac{\cancel{x}\left(3-\frac{4}{x}\right)}{-2\cancel{x}\sqrt{1+\frac{\pi}{4x}-\frac{3}{4x^2}}} = \lim\limits_{x\to -\infty} \dfrac{\left(3-\overbrace{\frac{4}{x}}^{0}\right)}{-2\sqrt{1+\underbrace{\frac{\pi}{4x}}_{0}-\underbrace{\frac{3}{4x^2}}_{0}}} = -\dfrac{3}{2}.$

2. The following parts are about calculation of derivatives. *(5×4pts)*

(A) Compute the following derivatives. (You do not need to simplify your answers.)

- $\frac{d}{dx}\left(\frac{x\sin x}{x^2-x-1}\right) = \frac{(x\cdot\sin x)'\cdot(x^2-x-1)-x\sin x\cdot(x^2-x-1)'}{(x^2-x-1)^2}$

$$= \frac{(\sin x + x\cdot\cos x)(x^2-x-1) - x\cdot\sin x(2x-1)}{(x^2-x-1)^2}$$

- $\frac{d}{dx}(x^2\sin 2x\cos 3x) = 2x\sin(2x)\cdot\cos(3x) + 2x^2\cos(2x)\cos(3x)$
$-3x^2\sin(2x)\sin(3x)$

- $\frac{d}{dx}\left(\sec(\sec(x^2))\right) = \sec(\sec(x^2))\cdot\tan(\sec(x^2))\cdot\sec(x^2)\cdot\tan(x^2)\cdot 2x$

(B) Write a formula for the second derivative $\frac{d^2}{dx^2}(f\circ g)(x)$ using f, g, f', g', f'', and g''.

$$\frac{d}{dx}(f\circ g)(x) \underset{\text{Chain Rule}}{=} f'(g(x))\cdot g'(x)$$

$$\frac{d^2}{dx^2}(f\circ g)(x) = \frac{d}{dx}\left(f'(g(x))\cdot g'(x)\right) = f''(g(x))\cdot[g'(x)]^2 + f'(g(x))\cdot g''(x)$$

(C) Compute the derivative $(f\circ g)'(1)$ given the following table of values:

x	0	1	2
$f(x)$	2	1	0
$g(x)$	1	2	0
$f'(x)$	2	0	1
$g'(x)$	0	1	2

$$(f\circ g)'(1) \underset{\text{Chain Rule}}{=} f'(g(1))\cdot g'(1) = f'(2)\cdot g'(1)$$

$$= 1\cdot 1 = 1.$$

3. *(8pts)* Approximate the value of $\cos\left(\sqrt[4]{17}\frac{\pi}{4}\right)$ using linear approximation.

We'll use linearization for $f(x) = \cos\left(\sqrt[4]{x}\cdot\frac{\pi}{4}\right)$ at $a = 16$.

$L(x) = f(16) + f'(16)\cdot(x-16)$ $\quad f'(x) = -\sin\left(\sqrt[4]{x}\cdot\frac{\pi}{4}\right)\cdot\frac{\pi}{4}\cdot\frac{1}{4}\cdot x^{-3/4}$

$f'(16) = -\sin\left(\sqrt[4]{16}\cdot\frac{\pi}{4}\right)\cdot\frac{\pi}{4}\cdot\frac{1}{4}\cdot(16)^{-3/4} = -\frac{\pi}{128}$

$L(17) = f(16) + f'(16)\cdot(17-16)$

$= 0 - \frac{\pi}{128}\cdot 1 = -\frac{\pi}{128}.$

4. *(12pts)* Find absolute maximum and absolute minimum of $f(x) = |4 - x^{2/3}|$ on $[-1, 27]$

$$f(x) = \begin{cases} 4 - x^{2/3} & x \in [-1, 8] \\ -(4 - x^{2/3}) & x \in (8, 27] \end{cases}$$

$\lim\limits_{x\to 8^+} f(x) = \lim\limits_{x\to 8^+} -(4 - x^{2/3}) = 0$

$\lim\limits_{x\to 8^-} f(x) = \lim\limits_{x\to 8^-} 4 - x^{2/3} = 0$

So, $f(x)$ is continuous on $[-1, 27]$. By Extreme Value Theorem, absolute maximum/minimum exist.

$$f'(x) = \begin{cases} -\frac{2}{3}\frac{1}{\sqrt[3]{x}} & x \in (-1, 8) \\ \frac{2}{3}\frac{1}{\sqrt[3]{x}} & x \in (8, 27) \end{cases}$$

$f(x)$ has critical points $x = 0$ and $x = 8$ ($f'(x)$ doesn't exist at 0 and 8)

Image of boundary Points:

$f(-1) = 4 - (-1)^{2/3} = 3$

$f(27) = -(4 - (27)^{2/3}) = 5 \longleftarrow$ Absolute Maximum Value

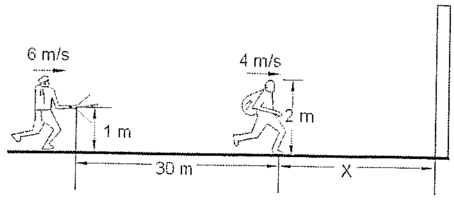
Image of Critical Points:

$f(0) = 4 - 0^{2/3} = 4$

$f(8) = 4 - 8^{2/3} = 0 \longleftarrow$ Absolute Minimum Value

5. *(14pts)* One dark night, a policeman (running 6 $\frac{\text{m}}{\text{s}}$) chases a 2 m tall robber (running 4 $\frac{\text{m}}{\text{s}}$) towords a wall. The policeman carries his torch 1 m above the ground.

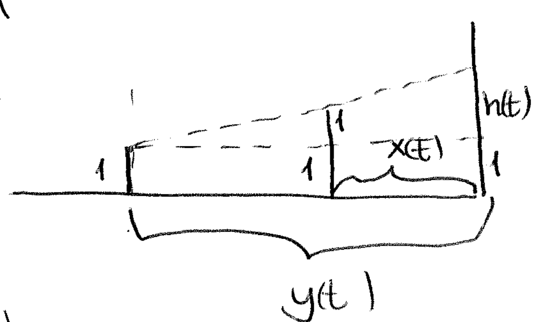
At the exact moment when the distance between the two is 30 m, the robber's shadow (cast on the wall by the policeman's torch) is not changing height. Assuming that they do not change speed, does the policeman catch the robber before he climbs the wall?



$y(t)$: distance of policeman to the wall

$x(t)$: distance of robber to the wall

$1+h(t)$: height of the shadow.



Using similar triangles, $\dfrac{1}{h(t)} = \dfrac{y(t)-x(t)}{y(t)}$

$$\Rightarrow h(t) = \frac{y(t)}{y(t)-x(t)}$$

By taking derivative $h'(t) = \dfrac{y'(t)\,(y(t)-x(t)) - (y'(t)-x'(t))\,y(t)}{(y(t)-x(t))^2}$

At that moment, let's say t_0; $y(t_0) - x(t_0) = 30$

$$y'(t_0) = -6 \quad \text{and} \quad x'(t_0) = -4$$

Since height is not changing $h'(t_0) = 0$.

$$0 = \frac{-6\,(30) - (-6-(-4))\,y(t_0)}{(30)^2}$$

$\Rightarrow y(t_0) = 90 \quad \Rightarrow x(t_0) = 90-30 = 60.$

Since $\dfrac{x(t_0)}{4} = \dfrac{y(t_0)}{6}$ so they meet at the wall.

6. *(18pts)* Graph the function $y = \dfrac{x^2+1}{(x-1)^2}$.
Find and label all asymptotes; intervals of increase and decrease; local maxima and minima; inflection points and concavity.

1) Domain $= \mathbb{R} \setminus \{1\}$

2) x-intercept: $\dfrac{x^2+1}{(x-1)^2}$ is never zero. No x-intercepts.

y-intercept: $y(0) = \dfrac{1}{1^2} = 1$ $(0,1)$ is the y-intercept.

3) Vertical Asymptotes: $\lim\limits_{x \to 1^+} \dfrac{x^2+1}{(x-1)^2} = +\infty$ $\lim\limits_{x \to 1^-} \dfrac{x^2+1}{(x-1)^2} = +\infty$

Horizontal Asymptotes: $\lim\limits_{x \to +\infty} \dfrac{x^2+1}{x^2-2x+1} = 1$ $\lim\limits_{x \to -\infty} \dfrac{x^2+1}{x^2-2x+1} = 1$

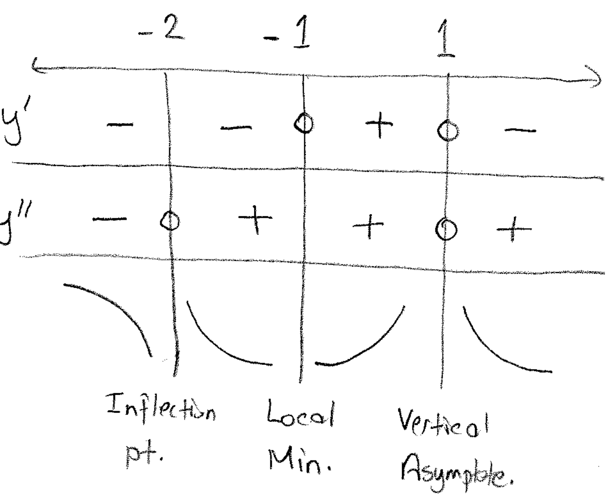
4) $y(-x) = \dfrac{(-x)^2+1}{(-x-1)^2} = \dfrac{x^2+1}{(x+1)^2}$ No Symmetry.

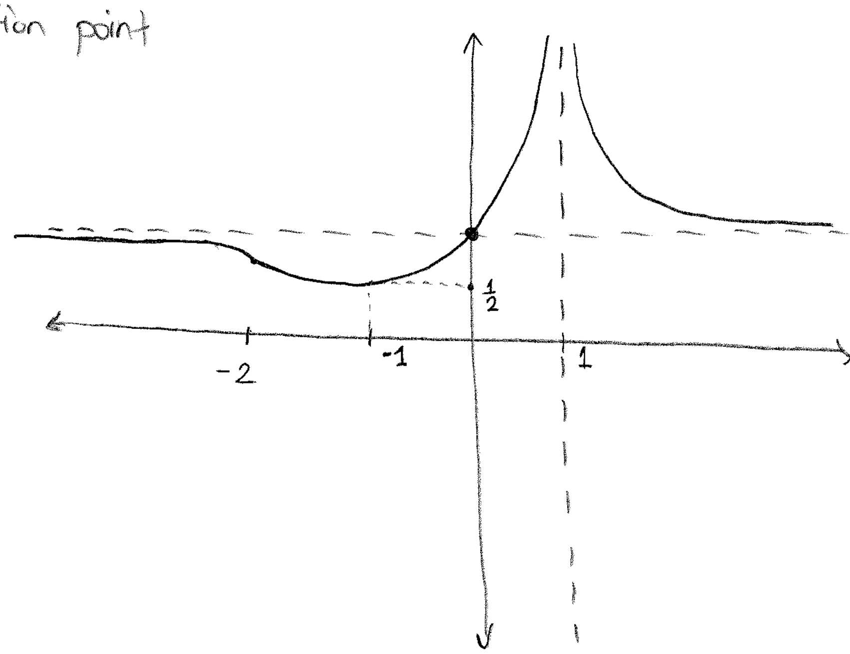
5) $y' = \dfrac{2x(x-1)^2 - (x^2+1)2(x-1)}{(x-1)^4} = \dfrac{2x^2-2x-2x^2-2}{(x-1)^3} = \dfrac{-2x-2}{(x-1)^3} = 0$

$x = -1$ is the only critical point.

6) $y'' = \dfrac{-2(x-1)^3 - (-2x-2)\,3(x-1)^2}{(x-1)^6} = \dfrac{-2x+2+6x+6}{(x-1)^4} = \dfrac{4x+8}{(x-1)^4} = 0$

$x = -2$ is a possible inflection point





7. *(5pts)* By using the definition of derivative, compute $f'(0)$ for $f(x) = \begin{cases} x^{3/2}\cos(\frac{\pi}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{3/2} \cdot \cos\left(\frac{\pi}{h}\right) - 0}{h} = \lim_{h \to 0} h^{1/2} \cdot \cos\left(\frac{\pi}{h}\right) = 0$$

$$-1 < \cos\left(\frac{\pi}{h}\right) < 1$$

$$-h^{1/2} < h^{1/2} \cdot \cos\left(\frac{\pi}{h}\right) < h^{1/2}$$ by Squeeze Theorem.

as $h \to 0$ as $h \to 0$

0

8. *(7pts)* Show that no function with $f''(x) > 0$ has three roots.
(Hint: Mean Value Theorem.)

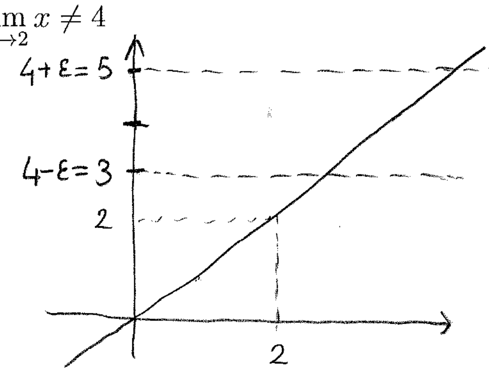
Suppose $f(x)$ three roots $a < b < c$. By applying Mean Value Thm. on intervals $[a,b]$ and $[b,c]$, there exist r in (a,b) and t in (b,c) such that $f'(r) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$, $f'(t) = \frac{f(b)-f(c)}{b-c} = \frac{0-0}{b-c} = 0$

Now, if we apply Mean Value Thm. on the interval $[r,t]$, for $f'(x)$, there exists s in (r,t) such that $f''(s) = \frac{f'(t) - f'(r)}{t-r} = \frac{0-0}{t-r} = 0$

Hence, we get a contradiction since $f''(x) > 0$.

Bonus Using the precise (ε-δ) definition of limit show that $\lim_{x \to 2} x \neq 4$

For $\varepsilon = 1$, for any $\delta > 0$, there exists x with $|x-2| < \delta$ but $|x-4| > \varepsilon = 1$.



For $\delta < 1$, we can choose $x = 2 + \frac{\delta}{2}$, and $|2 - \frac{\delta}{2} - 4| = |2 - \frac{\delta}{2}| > 1 = \varepsilon$

For $\delta \geq 1$, we can choose $x = \frac{5}{2}$, and $|\frac{5}{2} - 4| = \frac{3}{2} > 1 = \varepsilon$.