M E T U
Northern Cyprus Campus

	Introduction to Differential Equations Midterm I	

Code : *Math 219* Acad.Year: *2013-2014* Semester : *Fall* Date : *24.10.2013* Time : *17:40* Duration : *120 minutes*	Last Name: Name: KEY Department: Signature: Student No: Section: 5 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1 (20)	2 (20)	3 (20)	4 (20)	5 (20)		

Show your work! No calculators! Please draw a $\boxed{\text{box}}$ around your answers!

Please do not write on your desk!

1.(10+10=20 pts) (a) Find all solutions of the differential equation

$$ty' - y = t^2e^{-t} \Rightarrow y' - \frac{1}{t}y = te^{-t},\ t \neq 0$$

Integrating factor $\mu(t) = \frac{1}{|t|}$. Thereby

$\left(\frac{1}{t}\cdot y\right)' = e^{-t} \Rightarrow \frac{y}{t} = -e^{-t} + C \Rightarrow y = Ct - te^{-t}$

Thus $y = Ct - te^{-t}$ is the general solution.

(b) Suppose that a, b are real numbers and $a > 0$. Show that every solution of the differential equation

$$y' + ay = be^{-3t}$$

goes to 0 as $t \to \infty$. Again $\mu(t) = e^{at}$ is an integrating factor. We have $(e^{at}\cdot y)' = b\, e^{(a-3)t}$.

If $a \neq 3$ then $e^{at}\cdot y = \frac{b}{a-3} e^{(a-3)t} + C$ and

$y = Ce^{-at} + \frac{b}{a-3}e^{-3t}$ is the general solution and

$\lim_{t\to+\infty} y(t) = 0$

If $a = 3$ then $e^{at}\cdot y = bt + C$ and $y = Ce^{-at} + bte^{-at}$

or $y = Ce^{-3t} + bte^{-3t}$.

Note that $\lim_{t\to\infty} y(t) = 0$ by L'Hospital rule.

2.(20 pts) Solve the initial value problem

$$dx + (\frac{x}{y} - \sin y)dy = 0, \quad y(2) = \pi/2$$

by first finding an integrating factor of the form $\mu(y)$.

$\mu(y)\,dx + \mu(y)\left(\frac{x}{y} - \sin(y)\right)dy = 0$ has to be exact. By Component Test, we have

$$\mu'(y) = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\mu(y)}{y} \Rightarrow \frac{d\mu}{\mu} = \frac{dy}{y} \Rightarrow$$

$\mu(y) = y$ is an integrating factor.

Thus $y\,dx + (x - y\sin(y))\,dy = 0$ is an exact dif. equation, that is, there is a potential function $\Psi(x,y)$ such that

$$\begin{cases} \Psi_x = y & \Rightarrow \Psi = xy + C(y) \\ \Psi_y = x - y\sin(y) & = x + C'(y) \Rightarrow C'(y) = -y\sin(y) \end{cases}$$

It follows that

$$C(y) = -\int y\sin(y)\,dy = \int y\cos'(y)\,dy = y\cos(y)$$

$$-\int y'\cos(y)\,dy = y\cos(y) - \sin(y)$$

Hence $\Psi(x,y) = xy + y\cos(y) - \sin(y) = C$ is the general solution.

IVP: $x = 2,\ y = \pi/2 \Rightarrow \pi - 1 = C$

$$xy + y\cos(y) - \sin(y) = \pi - 1$$

is the solution to IVP in the implicit form.

3.(10+10=20 pts) (a) Show that the substitution $v = y^2/x$ converts any differential equation of the form

$$\frac{dy}{dx} = \frac{1}{y} f\left(\frac{y^2}{x}\right)$$

into a separable differential equation.

$$v = \frac{y^2}{x} \Rightarrow y^2 = xv \Rightarrow 2yy' = v + xv' \Rightarrow y' = \frac{v}{2y} + \frac{xv'}{2y}$$

So, $\frac{v}{2y} + \frac{xv'}{2y} = \frac{1}{y} f(v) \Rightarrow v + xv' = 2f(v)$ or

$x \frac{dv}{dx} = 2f(v) - v$ which is a separable differential equation w.r.t. v.

(b) Find all solutions of the equation

$$\frac{dy}{dx} = \frac{1}{y}\left(\frac{y^4}{x^2} - \frac{y^2}{2x}\right)$$

by using the method in part (a). Put $v = \frac{y^2}{x}$ and $f(v) = v^2 - \frac{v}{2}$.

Then $x \frac{dv}{dx} = 2v^2 - v - v \Rightarrow \frac{dv}{2(v^2 - v)} = \frac{dx}{x}$ $(v \neq 0, 1)$

$\Rightarrow \int \frac{dv}{v(v-1)} = \ln(C x^2)$, $C > 0 \Rightarrow \ln\left|\frac{v-1}{v}\right| = \ln(Cx^2)$

$\Rightarrow \left|\frac{v-1}{v}\right| = C x^2$, $C > 0 \Rightarrow \frac{v-1}{v} = C x^2$, $C \neq 0$

$\Rightarrow v = \frac{1}{1 - Cx^2}$, $C \neq 0 \Rightarrow y^2 = \frac{x}{1 - Cx^2}$, $C \neq 0$.

But $v \neq 0$, for $y \neq 0$.

$v = 1 \Rightarrow y^2 = x \Rightarrow y' = \frac{1}{2y} \Rightarrow y$ is a solution lost in the separation process.

So, $y^2 = \frac{x}{1 - Cx^2}$ is the general solution.

4.(8+8+4+5=25 pts) An object having an initial temperature of 25°C is placed in a medium (room, or a container filled with liquid, etc.) which has the same initial ambient temperature. The ambient temperature of the medium is raised linearly from 25°C to 30°C in 5 minutes (in other words, it is given by the function $25 + t$ for $0 \le t \le 5$). According to Newton's law of cooling, the rate of change of the temperature of the object is proportional to the difference between its temperature and the ambient temperature (with a proportionality constant $-k$ where $k > 0$ depends on the properties of the medium).

(a) Write an initial value problem that describes the situation.

$$\begin{cases} T'(t) = (-k)(T(t) - 25 - t) \\ T(0) = 25 \end{cases}$$

(b) Find the temperature $T(t)$ for $0 \le t \le 5$ in terms of k by solving the initial value problem in part (a).

$\mu(t) = e^{kt}$ is an integrating factor $\Rightarrow (e^{kt} T)' = k(25+t)e^{kt}$

$\Rightarrow e^{kt} \cdot T = 25e^{kt} + te^{kt} - \frac{1}{k}e^{kt} + C$, that is,

$T = Ce^{-kt} + (25 - \frac{1}{k} + t)$. But $T(0) = 25$, therefore

$25 = C + 25 - \frac{1}{k} \Rightarrow C = \frac{1}{k}$. Thus

$T(t) = \frac{1}{k}e^{-kt} + 25 - \frac{1}{k} + t$ is the solution to IVP.

$(0 \le t \le 5)$.

(c) Find $\lim\limits_{k \to 0+} T(5)$ and $\lim\limits_{k \to \infty} T(5)$. Since $T(5) = \dfrac{e^{-5k} - 1}{k} + 30$, we have

$\lim\limits_{k \to 0^+} T(5) = 30 - 5 = 25$ by L'Hospital rule.

$\lim\limits_{k \to \infty} T(5) = 30$.

(d) (Bonus) The case $k = 0$ is a medium with perfect isolation and the case $k = \infty$ is a medium with perfect heat conduction. How do your results in part (c) agree with your physical intuition?

$k = 0 \rightarrow$ the object ~~is~~ not effected by the temperature rise (no conduction)

$k = \infty \rightarrow$ the case of a perfect conduction.

5.(6+10+4=20 pts) Consider the initial value problem

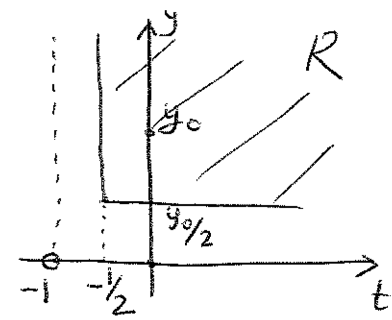
$$y' = \frac{t^4}{(1+t^5)y}, \quad y(0) = y_0$$

where $y_0 > 0$.

(a) Show that the conditions of the Existence and Uniqueness Theorem are satisfied, so this initial value problem has a unique solution.

For instance, put $R = [-\frac{1}{2}, +\infty) \times [\frac{y_0}{2}, +\infty)$.

Then $\frac{\partial f}{\partial y} = -\frac{t^4}{(1+t^5)y^2}$, and both $f, \frac{\partial f}{\partial y} \in C(R)$



(b) Find the solution of the initial value problem and determine its domain (the answer will depend on y_0).

$$\frac{dy}{dt} = \frac{t^4}{1+t^5}\frac{1}{y} \Rightarrow y\,dy = \frac{t^4\,dt}{1+t^5} \Rightarrow \frac{y^2}{2} = \frac{1}{5}\ln(1+t^5) + C$$

$(t > -1) \Rightarrow y^2 = \ln(1+t^5)^{2/5} + C$. But $y(0) = y_0$.

Therefore $C = y_0^2 \Rightarrow y^2 = \ln(1+t^5)^{2/5} + y_0^2 \Rightarrow$

$\Rightarrow y = \pm\left(\ln(1+t^5)^{2/5} + y_0^2\right)$. Since $y \geq 0$,

we have $y = \left(\ln(1+t^5)^{2/5} + y_0^2\right)^{1/2}$, and

$\ln(1+t^5)^{2/5} > -y_0^2 \Rightarrow 1+t^5 > e^{-\frac{5y_0^2}{2}} \Rightarrow t^5 > e^{-\frac{5}{2}y_0^2} - 1$

Thus $\mathrm{dom}(y) = \{t > (-1 + e^{-\frac{5}{2}y_0^2})^{1/5}\} \subseteq (-1, +\infty)$.

(c) Show that the domain of the solution is never $(-1, \infty)$ for any value of y_0.

Since $y_0 > 0$, it follows that

$-1 + e^{-\frac{5}{2}y_0^2} > -1 \Rightarrow (-1 + e^{-\frac{5}{2}y_0^2})^{1/5} > -1$

$\Rightarrow \mathrm{dom}(y) \subsetneq (-1, +\infty)$.