METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES FINAL EXAM	
Code : *MAT 120* Acad.Year: *2013-2014* Semester : *SPRING* Date : *04.06.2014* Time : *12:30* Duration : *120 min*	Last Name: Name : Solution Student # : Signature :
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (8)	2. (8)	3. (15)	4. (24)	5. (15)	6. (20)	7. (10)		

Please draw a box around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(8pts)* Give the equation of the tangent plane to $xz+2x^2y-yz^2=-11$ at the point (1, 2, 3).

Implicit tangent plane formula for $F(x,y,z)=k$:

$$F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0$$

$F(x,y,z) = xz+2x^2y-yz^2$ @ $(1,2,3)$

$F_x = z+4xy$ $\qquad F_x(1,2,3)=3+8=11$

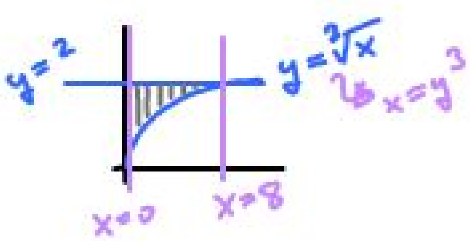
$F_y = 2x^2-z^2$ $\qquad F_y(1,2,3)=2-9=-7$

$F_z = x - 2yz$ $\qquad F_z(1,2,3)=1-12=-11$

$$\boxed{11(x-1)-7(y-2)-11(z-3)=0}$$

2. *(8pts)* Evaluate the iterated integral $\displaystyle\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$.

This is easier if we change the order of integration first...



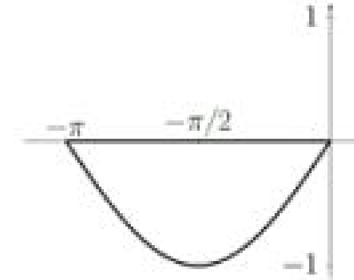
$$\int_{y=0}^{y=2}\int_{x=0}^{x=y^3}\frac{1}{y^4+1}\,dx\,dy = \int_{y=0}^{y=2}\frac{x}{y^4+1}\Big|_{x=0}^{x=y^3}dy$$

$$= \int_{y=0}^{y=2}\frac{y^3}{y^4+1}\,dy \qquad \left(\begin{matrix}u=y^4+1\\ du=4y^3\,dy\end{matrix}\right)$$

$$= \tfrac{1}{4}\ln|y^4+1|\Big|_{y=0}^{y=2}$$

$$= \tfrac{1}{4}(\ln 17-\ln 1) = \boxed{\begin{matrix}\tfrac{1}{4}\ln 17\\ \shortparallel \\ \ln\sqrt[4]{17}\end{matrix}}$$

3. *(15pts)* Evaluate the line integral

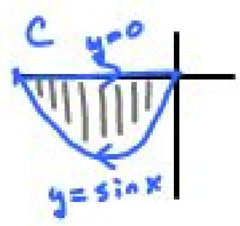


$$I = \oint_C (3y + \sqrt{x^4+1})\,dx \; + \; (2x - \sqrt{y^4+1})\,dy$$

where C is the closed curve which consists of the arc of the curve $y = \sin x$ from $(-\pi, 0)$ to $(0,0)$ and the line segment from $(0,0)$ to $(-\pi, 0)$.

$$\boxed{\underline{\text{Green's Thm}}: \quad \oint_C P\,dx + Q\,dy = \iint_{\text{inside}(C)} Q_x - P_y \, dA}$$

$$\oint_C (3y + \sqrt{x^4+1})\,dx + (2x - \sqrt{y^4+1})\,dy = \iint_{\text{inside}(C)} 2 - 3 \, dA$$



$$= \int_{x=-\pi}^{x=0} \int_{y=\sin x}^{y=0} -1 \; dy\,dx$$

$$= \int_{x=-\pi}^{x=0} -y \Big|_{\sin x}^{0} \, dx$$

$$= \int_{x=-\pi}^{x=0} \sin x \; dx$$

$$= -\cos x \Big|_{x=-\pi}^{x=0}$$

$$= -1 + (-1) = \boxed{-2}$$

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4. *(3×8=24pts)* Compute the power series for $f(x) = \dfrac{1}{(4-3x)^2}$ around $a = 0$

(a) Using the geometric series formula. $(\frac{1}{1-x} = \cdots)$

$$\frac{1}{(4-3x)^2} = \frac{d}{dx}\left(\frac{1}{3}\cdot\frac{1}{4-3x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{3}\cdot\frac{1}{4}\,\frac{1}{1-\frac{3}{4}x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{3\cdot 4}\cdot\sum_{n=0}^{\infty}\left(\tfrac{3}{4}x\right)^n\right) \qquad \left(\begin{array}{l}\text{for } \left|\tfrac{3}{4}x\right| < 1 \\ \text{i.e. } x < \left|\tfrac{4}{3}\right|\end{array}\right)$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{3\cdot 4}\cdot\frac{3^n}{4^n}x^n$$

$$= \boxed{\sum_{n=1}^{\infty}\frac{3^{n-1}}{4^{n+1}}\,n\,x^{n-1} \underset{\text{reindex}}{=} \sum_{n=0}^{\infty}\frac{3^n}{4^{n+2}}(n+1)\,x^n}$$

(b) Using Taylor's series formula.

$f(x) = \dfrac{1}{(4-3x)^2}$ $\qquad f(0) = 4^{-2}$

$f'(x) = -2(-3)(4-3x)^{-3}$ $\qquad f'(0) = 2\cdot 3\cdot 4^{-3}$

$f''(x) = (-2)(-3)\cdot(-3)^2(4-3x)^{-4}$ $\qquad f''(0) = 3!\cdot 3^2\cdot 4^{-4}$

$f'''(x) = (-2)(-3)(-4)\cdot(-3)^3(4-3x)^{-5}$ $\qquad f'''(0) = 4!\cdot 3^3\cdot 4^{-5}$

$\vdots \qquad\qquad \vdots$

$$f^{(n)}(0) = (n+1)!\;3^n\,4^{-(n+2)}$$

Taylor's Formula:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(n+1)!\,3^n\,4^{-(n+2)}}{n!}x^n = \boxed{\sum_{n=0}^{\infty}\frac{(n+1)\cdot 3^n}{4^{n+2}}x^n}$$

(c) Using the binomial series formula.

Note: $\displaystyle\binom{-2}{n} = \frac{(-2)(-3)\cdots(-2-n+1)}{n!}$

$\displaystyle = (-1)^n\frac{(n+1)!}{n!} = \underline{\underline{(-1)^n(n+1)}}$

$$\frac{1}{(4-3x)^2} = (4-3x)^{-2}$$

$$= 4^{-2}\cdot\left(1-\tfrac{3}{4}x\right)^{-2}$$

$$= \frac{1}{4^2}\cdot\left(1+\left(-\tfrac{3}{4}x\right)\right)^{-2}$$

$$= \frac{1}{4^2}\sum_{n=0}^{\infty}\binom{-2}{n}\left(-\tfrac{3}{4}x\right)^n = \boxed{\sum_{n=0}^{\infty}\binom{-2}{n}(-1)^n\frac{3^n}{4^{n+2}}x^n}$$

5. *(5×3=15pts)* Determine whether the given series are convergent or divergent. State which tests you use, and give details.

(a) $\sum_{n=9}^{\infty} (\sqrt[n]{2}-1)^n$

Root Test: $\lim \left(\left(\sqrt[n]{2}-1\right)^n\right)^{1/n} = \lim \sqrt[n]{2} - 1 = 0$

$\hookrightarrow \lim \sqrt[n]{2} = e^{\lim \ln \sqrt[n]{2}} = e^{\lim \frac{1}{n}\ln 2} = e^0 = 1$

Since $\lim (|a_n|)^{1/n} = 0 < 1$

the sequence is absolutely convergent

(b) $\sum_{n=7}^{\infty} \frac{\ln n}{n^2}$

Comparison Test: $0 < 1 < \ln n < n^{1/2}$ for n big

$\hookrightarrow 0 < \frac{1}{n^2} < \frac{\ln n}{n^2} < \frac{n^{1/2}}{n^2}$

$\hookrightarrow \sum \frac{1}{n^2}$ is a convergent p-series $(p=2)$ $\Rightarrow$ NOT USEFUL

$\hookrightarrow \sum \frac{n^{1/2}}{n^2} = \sum \frac{1}{n^{3/2}}$ is a convergent p-series $(p=3/2)$ $\Rightarrow$ USEFUL.

So $\sum \frac{\ln n}{n^2}$ is convergent by the comparison test with the convergent p-series $\sum \frac{n^{1/2}}{n^2}$

(c) $\sum_{n=2}^{\infty} \cos\left(\frac{1}{n}\right)$

Test for Divergence (a.k.a. "n^{th} Term Test"):

$\lim \cos(1/n) = \cos(0) = 1$

But $1 \neq 0$ so $\sum \cos(1/n)$ is divergent.

(d) $\sum_{n=3}^{\infty} \frac{\sqrt{1+n^3}}{n^2+2n+1}$

Limit Comparism Test (with $\sum \frac{\sqrt{n^3}}{n^2} = \sum \frac{1}{n^{2-3/2}} = \sum \frac{1}{n^{1/2}}$)

$$\lim \frac{\frac{\sqrt{1+n^3}}{n^2+2n+1}}{\frac{\sqrt{n^3}}{n^2}} = \lim \frac{\sqrt{\frac{1+n^3}{n^3}}}{\frac{n^2+2n+1}{n^2}} = \lim \frac{\sqrt{1/n^3+1}}{1+2/n+1/n^2} = \frac{1}{1} \neq 0, \neq \infty$$

So the series behaves like $\sum \frac{\sqrt{n^3}}{n^2} = \sum \frac{1}{n^{1/2}}$ a divergent p-series $(p=\frac{1}{2})$

(e) $\sum_{n=5}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

Alternating series test:

$\lim \frac{\sqrt{n}}{n+1} = \lim \frac{\sqrt{n}}{n} \frac{1}{1+1/n} = 0$

and

$\frac{\sqrt{n}}{n+1}$ is decreasing

So the series is convergent by the alternating series test.

$\hookrightarrow$ If $f(x) = \frac{\sqrt{x}}{x+1}$ then $f' = \frac{x+1-2x}{2\sqrt{x}(x+1)^2} < 0$ for $x > 1$

6. *(20pts)* Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=3}^{\infty} (-1)^n \frac{(2x-3)^n \ln n}{(3n+1)\,5^n} = \sum (-1)^n \frac{2^n \cdot (x-3/2)^n \cdot \ln n}{(3n+1)\cdot 5^n}$$

$\rightsquigarrow$ center $= 3/2$.

Ratio Test: Convergent if

$$1 > \lim \left| \frac{\dfrac{(-1)^{n+1} \cdot 2^{n+1} \cdot (x-3/2)^{n+1} \cdot \ln(n+1)}{(3(n+1)+1)\cdot 5^{n+1}}}{\dfrac{(-1)^n \cdot 2^n \cdot (x-3/2)^n \cdot \ln n}{(3n+1)\cdot 5^n}} \right| = \lim \left| \frac{\dfrac{2^{n+1}}{2^n} \cdot \dfrac{\ln(n+1)}{\ln n} \cdot \dfrac{(x-3/2)^{n+1}}{(x-3/2)^n}}{\dfrac{(3n+4)}{3n+1} \cdot \dfrac{5^{n+1}}{5^n}} \right| = \frac{2}{5}\left|x - 3/2\right|$$

$\left(\text{Note: } \lim \frac{\ln(n+1)}{\ln n} \rightsquigarrow \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1 \right)$

Convergent if

$1 > 2/5\,|x - 3/2|$

$5/2 > |x - 3/2|$

$$\boxed{\text{Radius of convergence} = 5/2}$$

Check endpoints:

If $x = 3/2 - 5/2$ then the series is:

$$\sum (-1)^n \frac{2^n \cdot (-5/2)^n \cdot \ln n}{(3n+1)\cdot 5^n} = \sum \underbrace{(-1)^{2n}}_{=1} \frac{\ln n}{3n+1}$$

This is divergent by limit comparison with $\sum \frac{\ln n}{3n}$

$\left(\lim \frac{\frac{\ln n}{3n+1}}{\frac{\ln n}{3n}} = \lim \frac{\frac{\ln n}{\ln n}}{\frac{3n+1}{3n}} = \frac{1}{1} \; \frac{*^{0}}{*_{\infty}} \right)$

and then comparison with $\sum 1/3n$

$\left(0 < 1 < \ln n \rightsquigarrow 0 < 1/3n < \ln n/3n \right.$

$\left. \text{but } \sum 1/3n \text{ is divergent p-series} \right)$

If $x = 3/2 + 5/2$ then the series is:

$$\sum (-1)^n \frac{2^n (5/2)^n \ln n}{(3n+1)\cdot 5^n} = \sum (-1)^n \frac{\ln n}{3n+1}$$

This is convergent by the alternating series test.

$\frac{\ln n}{3n+1} > 0$ for $n > 1$

with $\lim \frac{\ln n}{3n+1} \rightsquigarrow \lim_{x\to\infty} \frac{\ln x}{3x+1} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{3} = 0$

and $f(x) = \frac{\ln x}{3x+1}$ has $f'(x) = \frac{3x+1-3x\ln x}{x(3x+1)^2} < 0$ for $x > e^2$

so $\frac{\ln n}{3n+1}$ is decreasing

$$\boxed{\text{Interval of convergence: } (3/2 - 5/2,\ 3/2 + 5/2] \quad \text{i.e. } -1 < x \leq 4}$$

7. *(8+2=10pts)* **(a)** Write a series which converges to $\displaystyle\int_0^{1/2} \arctan(x^2)\,dx$.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (\text{for } |x|<1)$$

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} \qquad (\text{for } |x^2|<1 \rightsquigarrow |x|<1)$$

$$\int_0^{1/2} \arctan(x^2)\,dx = \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}\,dx = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2n+1} \int_0^{1/2} x^{4n+2}\,dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{4n+3}}{4n+3} \Big|_0^{1/2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} \cdot \frac{1}{2^{4n+3}}}$$

(b) Will this method work to get a series for $\displaystyle\int_0^{2} \arctan(x^2)\,dx$??
Why or why not?

No. The power series expansion

$$\arctan(x^2) = \sum (-1)^n \frac{x^{4n+2}}{2n+1}$$

is only valid for $|x|<1$

But to evaluate $\int_0^2 \cdots dx$ you would need $x=2$ which makes the series divergent!!

Note that $\int_0^2 \arctan(x^2)\,dx$ is a number (≈ 1.418)...

You just have to compute it some other way.

$\rightsquigarrow$ You could do a long, messy integration by parts and partial fractions... or maybe use power series centered at 1.