METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 2	
Code : *MAT 120*	Last Name:
Acad.Year: *2012-2013*	Name : Student No.:
Semester : *FALL*	Department: Section:
Date : *22.12.2012*	Signature:
Time : *14:40*	5 QUESTIONS ON 6 PAGES
Duration : *110 minutes*	TOTAL 100 POINTS

1. (18)	2. (20)	3. (21)	4. (21)	5. (20)	Bonus		

Show your work! Please draw a $\boxed{\text{box}}$ around your answers!

1. *(3× 7pts)* Compute the following double integrals.

(a) $\displaystyle\int_0^1\int_{-1}^{2} \frac{x}{y+1}\,dx\,dy$

$$\int_0^1 \tfrac{1}{2}x^2 \frac{1}{y+1}\Big|_{x=-1}^{x=2} dy = \int_0^1 \left(2-\tfrac{1}{2}\right)\frac{1}{y+1}\,dy$$

$$= \tfrac{3}{2}\ln|y+1| \Big|_{y=0}^{y=1}$$

$$= \tfrac{3}{2}(\ln 2 - \ln 1) = \boxed{\tfrac{3}{2}\ln 2}$$

(b) $\displaystyle\int_1^2\int_0^{\sqrt{x}} y\sqrt{x^2+1}\,dy\,dx$

$$\int_1^2 \tfrac{1}{2}y^2\sqrt{x^2+1}\Big|_{y=0}^{y=\sqrt{x}} dx = \int_1^2 \tfrac{1}{2}|x|\sqrt{x^2+1}\,dx$$

integrating only for positive x, so $|x| = x$.

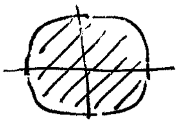
$u = x^2+1$
$du = 2x\,dx$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{2}{3}(x^2+1)^{3/2}\Big|_{x=1}^{x=2}$$

$$= \boxed{\tfrac{1}{6}\left(5^{3/2} - 2^{3/2}\right)}$$

(c) $\displaystyle\iint_R y^3\,dA$ where R is the region enclosed by $x^4+y^4=1$.

Integral is either $\displaystyle\int_{-1}^{1}\int_{y=-\sqrt[4]{1-x^4}}^{y=\sqrt[4]{1-x^4}} y^3\,dy\,dx$ or $\displaystyle\int_{-1}^{1}\int_{x=-\sqrt[4]{1-y^4}}^{x=\sqrt[4]{1-y^4}} y^3\,dx\,dy$



$x^4+y^4=1$

$$\int_{-1}^{1}\int_{y=-\sqrt[4]{1-x^4}}^{y=\sqrt[4]{1-x^4}} y^3\,dy\,dx = \int_{-1}^{1} \tfrac{1}{4}y^4\Big|_{y=-\sqrt[4]{1-x^4}}^{y=\sqrt[4]{1-x^4}} dx = \int_{-1}^{1} 0\,dx = \boxed{0}$$

$$\int_{-1}^{1}\int_{x=-\sqrt[4]{1-y^4}}^{x=\sqrt[4]{1-y^4}} y^3\,dx\,dy = \int_{-1}^{1} xy^3\Big|_{x=-\sqrt[4]{1-y^4}}^{x=\sqrt[4]{1-y^4}} dy = \int 2y^3\sqrt[4]{1-y^4}\,dy$$

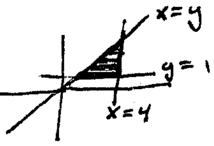
$$= 2\left(-\tfrac{1}{4}\right)\cdot\tfrac{4}{5}(1-y^4)^{5/4}\Big|_{y=-1}^{y=1} = \boxed{0}$$

Note! $f(x,y) = y^3$ is an odd function on a symmetric region, so $\iint f\,dA = 0$

2. *(4× 5pts)* In the following parts, perform the indicated changes to the integral, **but do NOT integrate the result.**

(a) For the following integral, change the order of integration to $dy\,dx$ (do NOT integrate).

$$\int_1^4 \int_y^4 \sin(x^2)\;dx\,dy$$

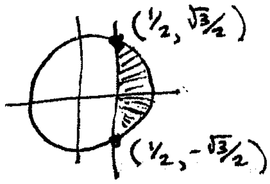


Region of int.

$$\boxed{\int_1^4 \int_1^x \sin(x^2)\;dy\,dx}$$

(b) Convert the following integral to polar coordinates (do NOT integrate).

$\iint_R x + y^2\; dA$ where R is the region between $x = \frac{1}{2}$ and $x^2 + y^2 = 1$.



Region of int.

In polar coords. • $x = \frac{1}{2} \rightsquigarrow r\cos\theta = \frac{1}{2}$

$r = \frac{1}{2}\sec\theta$

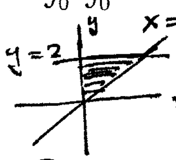
• $x^2 + y^2 = 1 \rightsquigarrow r = 1$

• point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is at angle $\theta = \frac{\pi}{3}$

$$\boxed{\int_{-\pi/3}^{\pi/3} \int_{r=\frac{1}{2}\sec\theta}^{r=1} (r\cos\theta + r^2\sin^2\theta)\, r\,dr\,d\theta}$$

(c) Change coordinates to s, t where $x = s+t$ and $y = s-t$ in the following (do NOT integrate).

$$\int_0^2 \int_0^y \sin(x-y)\, e^{x+y}\; dx\,dy.$$



Region of int in (x,y)

Boundary:

• $x = y \rightsquigarrow s+t = s-t$

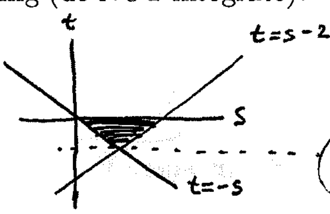
$0 = t$

• $y = 2 \rightsquigarrow s - t = 2$

$s = t+2$

• $x = 0 \rightsquigarrow s+t = 0$

$s = -t$



$-t = t+2$
$\Downarrow$
$t = -1$

Region of int in (s,t)

Jacobian:

$$\frac{\partial(x,y)}{\partial(s,t)} = 1\cdot(-1) - 1\cdot 1 = -2$$

$$\boxed{\int_{-1}^{0} \int_{-t}^{t+2} \sin(2t)\, e^{2s} \cdot 2\; ds\,dt}$$

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3. *(4× 5pts)* Convert the following line integrals to integrals of t, but **do NOT integrate**.

(a) $\int_C x^2 + \sin(xy)\, ds$ where $C = \{\langle \sin(t),\ \cos(2t)\rangle,\ \text{for } 0 \leqslant t \leqslant \pi\}$.

$x = \sin t$
$y = \cos 2t$

$ds = \sqrt{\cos^2 t + 4\sin^2 2t}\, dt$

+1 +1 +3

$$\boxed{\int_0^\pi \left(\sin^2 t + \sin(\sin t \cdot \cos 2t)\right)\sqrt{\cos^2 t + 4\sin^2 2t}\, dt}$$

(b) $\int_C (x^2 + y)\, dx + (y^2 - x)\, dy$ where C is the line segment from $(1, 2)$ to $(3, 5)$.

Parameterization of C: $\vec{r}(t) = (1, 2) + (2, 3)t$

$\begin{cases} x = 1 + 2t \rightsquigarrow dx = 2\,dt \\ y = 2 + 3t \rightsquigarrow dy = 3\,dt \end{cases}$

Alternate parameterization: $\vec{r} = \langle 1, 2/3\rangle + \langle 0, 1/2\rangle$

$\begin{cases} x = t \\ y = 2/3\, t + 1/2 \end{cases}$

+1 +1 +1 +1 +1

$$\boxed{\int_0^1 \left((1+2t)^2 + (2+3t)\right)(2) + \left((2+3t)^2 - (1+2t)\right)(3)\, dt}$$

$\int_0^1 35t^2 + 44t + 15\, dt$

(c) $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = \langle x,\ y - x\rangle$ and C is $\mathbf{r}(t) = \langle e^t, t^2\rangle$ for $0 \leqslant t \leqslant 1$.

$\begin{cases} x = e^t \rightsquigarrow dx = e^t\,dt \\ y = t^2 \rightsquigarrow dy = 2t\,dt \end{cases}$

+1 +1 +1 +1 +1

$$\boxed{\int_0^1 e^t \cdot e^t + (t^2 - e^t) \cdot 2t\, dt} = \int_0^1 e^{2t} + 2t^3 - 2te^t\, dt$$

(d) $\oint_C \mathbf{F} \bullet \mathbf{T}\, ds$ where $\mathbf{F} = \langle y, 0\rangle$ and C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ oriented clockwise. +1

(Recall: $\mathbf{T}(x, y)$ is the tangent vector to C at (x, y).)

Parameterization of C: $x = 2\cos t$
$y = -3\sin t$ +1

+1 +1 +1

$$\boxed{\int_0^{2\pi} -3\sin t \cdot (-2\sin t)\, dt}$$

4. *(7+7+4pts)* The following parts each use the answer of the preceeding part. If you give an incorrect answer, then no further parts will be graded.

(a) Check each of the vector fields below and find the one which is conservative (**exactly one** is conservative; to receive credit you must check them all).

$$\mathbf{A} = \left\langle x\ln(x+y),\ x - y\ln(x+y)\right\rangle, \quad \mathbf{B} = \left\langle xy,\ -\frac{1}{2}x^2\right\rangle, \quad \mathbf{C} = \left\langle y\sin(xy),\ x\sin(xy) + 1\right\rangle$$

$\mathbf{A}$: $\partial/\partial y$: $\dfrac{x}{x+y}$, $\quad\partial/\partial x$: $1 - \dfrac{y}{x+y} = \dfrac{x}{x+y}$

These are equal, but not continuous at $(0,0)$ so this is not conservative!

$\mathbf{B}$: $\partial/\partial y$: x, $\quad\partial/\partial x$: $-x$, $\quad x \neq -x$

Not conservative

$\mathbf{C}$: $\partial/\partial y$: $\sin(xy) - xy\cos(xy)$, $\quad\partial/\partial x$: $= \sin(xy) - xy\cos(xy)$

This vector field is conservative!

$$\boxed{\langle y\sin(xy),\ x\sin(xy) + 1\rangle}$$

(b) Find the potential function for the conservative vector field from (a).

$$\int y\sin(xy)\,dx = y\cdot\frac{1}{y}\left(-\cos(xy)\right) + C_y$$

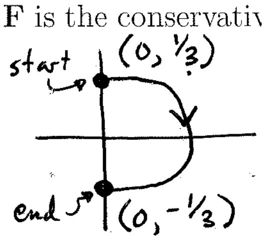
$$= -\cos(xy) + C_y$$

$$\partial/\partial y \; -\cos(xy) = x\sin(xy)$$

$$\int x\sin(xy) + 1 - x\sin(xy)\,dy = \int 1\,dy = y$$

$$\boxed{f = -\cos(xy) + y}$$

(c) Use your answer from (b) to compute the line integral $\int_C \mathbf{F}\bullet d\mathbf{r}$ where C is the path around the right half of the ellipse $4x^2 + 9y^2 = 1$ in the **clockwise** direction and $\mathbf{F}$ is the conservative vector field from (a).



$$\int_C \vec{F}\cdot d\vec{r} = f(\text{end}) - f(\text{start})$$

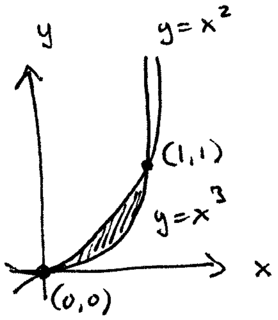
$$= -\cos\left(0\cdot\left(-\tfrac{1}{3}\right)\right) + \left(-\tfrac{1}{3}\right)$$

$$-\left(-\cos\left(0\cdot\left(\tfrac{1}{3}\right)\right) + \tfrac{1}{3}\right) = \boxed{-\tfrac{2}{3}}$$

5. *(20pts)* Let $R = \{(x, y) : x^3 \leqslant y \leqslant x^2\}$. Compute the following integral, using Green's theorem or otherwise,

$$\int_C \mathbf{F} \bullet d\mathbf{r},$$

where $\mathbf{F} = x^2\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$ and C is a counterclockwise oriented boundary of R.



$y = x^2$ and $y = x^3$ cross at

$x^2 = x^3$

$0 = x^2(x-1)$

$x = 0, 1$

We will use Green's Thm:

$$\oint_C x^2\,dx + xy^2\,dy = \iint_{\text{inside } C} \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x^2)\,dA$$

$$= \int_0^1 \int_{x^3}^{x^2} y^2 - 0 \;dy\,dx$$

$$= \int_0^1 \frac{1}{3} y^3 \Big|_{y=x^3}^{y=x^2} dx$$

$$= \int_0^1 \frac{1}{3}x^6 - \frac{1}{3}x^9 \;dx$$

$$= \frac{1}{21}x^7 - \frac{1}{30}x^{10} \Big|_{x=0}^{x=1}$$

$$= \frac{1}{21} - \frac{1}{30} = \boxed{\frac{1}{70}}$$

Bonus. Calculate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. (The answer is not ∞.)

(Hint: First multiply by $\int_{-\infty}^{\infty} e^{-y^2}\,dy$ to get a double integral.)

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \left(\int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2}\,dx\,dy \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \right)^{1/2}$$

— change to polar coordinates —

$$= \left(\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \right)^{1/2}$$

$$= \left(\int_{0}^{2\pi} -\tfrac{1}{2} e^{-r^2} \Big|_{r=0}^{r=\infty} d\theta \right)^{1/2}$$

$$= \left(\int_{0}^{2\pi} \tfrac{1}{2}\,d\theta \right)^{1/2}$$

$$= \boxed{\sqrt{\pi}}$$