METU – NCC
Mathematics Group

	Calculus with Analytic Geometry Midterm Exam	
Code : MAT 120 Acad.Year : 2011-2012 Semester : Fall Instructors : Anar Dosiev Date : 28.11.2011 Time : 17.40 Duration : 100 minutes	Last Name : Name : KEY Dept. : Signature :	Stud. No : Sec. No : 7 Questions on 4 Pages Total 100 Points

1 (10)	2 (10)	3 (15)	4 (15)	5 (15)	6 (20)	7 (15)	

Q.1 (10 pts) Evaluate the following limit: $\lim_{(x,y)\to(0,0)} \frac{\sin(y)\,x^3}{y\,(x^2+3y^2)}$. Explain your answer.

Let's prove that $\lim_{(x,y)\to(0,0)} \frac{\sin(y)x^3}{y(x^2+3y^2)} = 0$. Fix $\varepsilon > 0$ and put $\delta = \min\{\frac{\varepsilon}{2}, 1\}$. For each (x,y) with $\|(x,y)\| \le \delta$ we have

$$\left|\frac{\sin(y)\,x^3}{y(x^2+3y^2)}\right| = \left|\frac{\sin(y)}{y}\right| |x| \frac{x^2}{x^2+3y^2} \le 2\,|x|\cdot 1 =$$

$$= 2\sqrt{x^2} \le 2\sqrt{x^2+y^2} = 2\,\|(x,y)\| \le 2\delta$$

$$\le \varepsilon.$$

Q.2 (10 pts) Suppose we know that the cross product $\mathbf{a}\times\mathbf{b} = 2\mathbf{j}$ for some vectors $\mathbf{a}$ and $\mathbf{b}$ in space. Find the number $(((\mathbf{a}-7\mathbf{b})\times(\mathbf{a}+4\mathbf{b}))\times\mathbf{i})\cdot\mathbf{k}$, where $\mathbf{i},\mathbf{j},\mathbf{k}$ are the unit basis vectors in space.

Based on the rules of the cross-product, we have $(\vec{a}-7\vec{b})\times(\vec{a}+4\vec{b}) = 4\vec{a}\times\vec{b} - 7\vec{b}\times\vec{a} = 4\vec{a}\times\vec{b} + 7\vec{a}\times\vec{b} = 11\vec{a}\times\vec{b}$. Therefore

$$((11\,\vec{a}\times\vec{b})\times\vec{i})\cdot\vec{k} = 22(\vec{j}\times\vec{i})\cdot k = -22\,\vec{k}\cdot\vec{k}$$

$$= -22$$

Q.3 (9+4+2=15 pts) Consider the following lines $x = 3t$, $y = 1+2t$, $z = -1-t$, and $x = 3+s$, $y = 3-s$, $z = -2+4s$ in space.

(a) Find their intersection point and write down the equation of the plane in space generated by these lines.

First we put $3t = 1+s$, $1+2t = 3-s$, $-1-t = -2+4s$, whose solution: $t=1$, $s=0 \Rightarrow P(3,3,-2)$ is the intersection point of the lines. Consider the vectors $\vec{v}_1 = \langle 3,2,-1\rangle$, $\vec{v}_2 = \langle 1,-1,4\rangle$ of the lines.

Then $\vec{u} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 1 & -1 & 4 \end{vmatrix} = 7\vec{i} - 13\vec{j} - 5\vec{k}$

is the normal vector to the plane sought

Whence $7(x-3) - 13(y-3) - 5(z+2) = 0$ or

$M: 5z = 7x - 13y + 8$

is the equation of the plane

(b) Find the equation of the line through the origin which is perpendicular to the plane from the previous item (a).

The vector $\vec{u}$ must generate this line

Thus

$$\ell: \begin{cases} x = 7t \\ y = -13t \\ z = -5t \end{cases}$$

is the line through the origin parallel to $\vec{u}$

(c) Find the distance from the origin up to the plane and show that it exceeds 2.

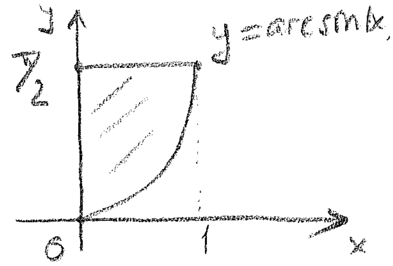
Let's find out the point $Q = \ell \cap M$:

$-25t = 49t + 169t + 8 \Rightarrow t = -\frac{8}{243}$

$\Rightarrow Q = \left(-\frac{8\cdot 7}{243}, \frac{13\cdot 8}{243}, \frac{5\cdot 8}{243}\right) \Rightarrow d = \frac{8}{\sqrt{243}}$

Q.4 (15 pts) Reverse the intergration order and compute the following double integral $\int_0^1 \left(\int_{\arcsin(x)}^{\pi/2} 2\cos(y)\sqrt{1+\cos^2(y)}dy\right)dx$. Sketch the integration region.

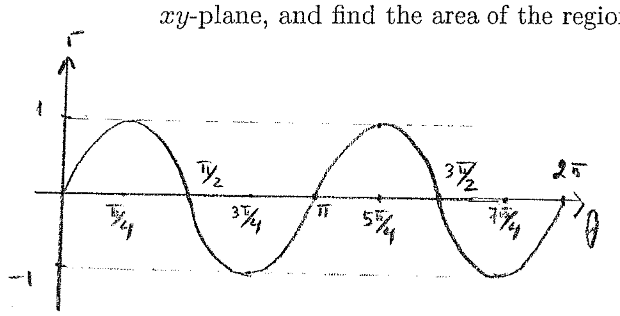
We have $I = \int_0^{\pi/2}\left(\int_0^{\sin(y)} 2\cos(y)\sqrt{1+\cos^2(y)}\,dx\right)dy$

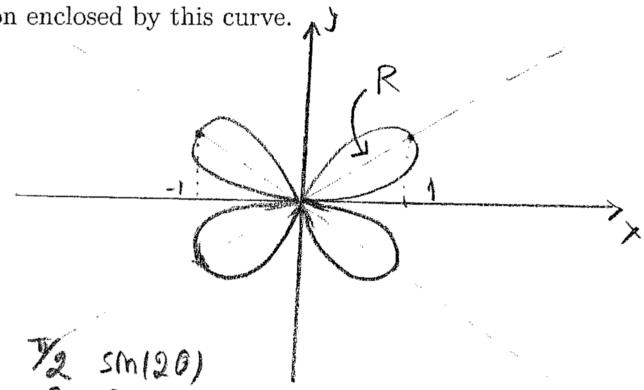


$= \int_0^{\pi/2} 2\cos(y)\sqrt{1+\cos^2(y)}\,\sin(y)\,dy =$

$= \left| \begin{array}{l} u = 1+\cos^2(y) \\ du = -2\cos(y)\sin(y)\,dy \\ y=0 \Rightarrow u=2 \\ y=\pi/2 \Rightarrow u=1 \end{array} \right| = \int_1^2 \sqrt{u}\,du = \frac{2}{3}\left(2^{3/2}-1\right)$

Q.5 (15 pts) Sketch the polar curve $r = \sin(2\theta)$ first in θr-plane, then in the xy-plane, and find the area of the region enclosed by this curve.





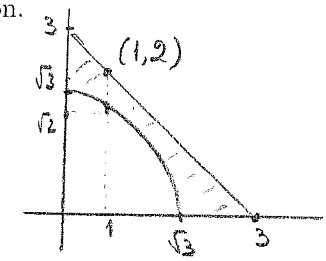
the area $= 4\iint_R dA = 4\int_0^{\pi/2}\int_0^{\sin(2\theta)} r\,dr\,d\theta =$

$= 4\int_0^{\pi/2}\frac{1}{2}\sin^2(2\theta)\,d\theta = 2\int_0^{\pi/2}\sin^2(2\theta)\,d\theta =$

$= \int_0^{\pi/2}(1-\cos(4\theta))\,d\theta = \frac{\pi}{2} - \frac{1}{4}\sin(4\theta)\Big|_0^{\pi/2} =$

$= \frac{\pi}{2}$

Q.6 (20 pts) Find the absolute maximum and minimum values of the function $f(x,y) = xy^2$ on the region $D = \{(x,y) : x \geq 0, y \geq 0, x^2 + y^2 \geq 3, y \leq 3 - x\}$ (Don't use Lagrange multipliers for the boundary). Sketch the region.



Note that $f_x = y^2, f_y = 2xy \Rightarrow$ no critical point inside of the region

On the boundary:

1) $x^2+y^2=3 \Rightarrow g(x)=f(x,\sqrt{3-x^2})=x(3-x^2)=$
$=-x^3+3x,\ 0\leq x\leq \sqrt{3};\ g'(x)=-3x^2+3=0 \Rightarrow x=1, y=\sqrt{2}$

2) $y=3-x \Rightarrow g(x)=f(x,3-x)=x(3-x)^2=x^3-6x^2+9x,\ 0\leq x\leq 3;$
$g'(x)=3x^2-12x+9=0$ or $x^2-4x+3=0 \Rightarrow x=1,3 \Rightarrow$
$(1,2), (3,0)$.

Hence we have got the following points:
$(1,\sqrt{2}), (1,2), (x,0), (0,y), \sqrt{3}\leq x,y\leq 3.$

$f(1,\sqrt{2})=2,\ f(1,2)=4,\ f(x,0)=f(0,y)=0,\ \forall x,y.$

$f(1,2)=4$: abs. max, $f(x,0)=f(0,y)=0$: abs. min.

Q.7 (15 pts) Using Lagrange multipliers for the boundary, find the absolute maximum and minimum values of the function $f(x,y) = x^2 + (y+1)^2$ on the region $x^2 + y^2 \leq 1$. Sketch the surface based on shifting techniques, and show us the relevant max and min-values.

$\nabla f = 2x\vec{i} + 2(y+1)\vec{j},\ \nabla g = 2x\vec{i} + 2y\vec{j},\ g(x,y)=x^2+y^2.$

No critical points inside of the region.

$$\begin{cases} x=\lambda x \\ y+1=\lambda y \\ x^2+y^2=1 \end{cases}$$

$\lambda=0 \Rightarrow x=0, y=-1 \Rightarrow (0,-1)$

$\lambda\neq 0 \Rightarrow$ if $x=0 \Rightarrow y=1$ (with $\lambda=2$) $\Rightarrow (0,1)$

if $x\neq 0 \Rightarrow \lambda=1 \Rightarrow y+1=y$, a contradiction.

$f(0,-1)=0$ (abs. min), $f(0,1)=4$ (abs. max)