PROOF We split the integral in two:

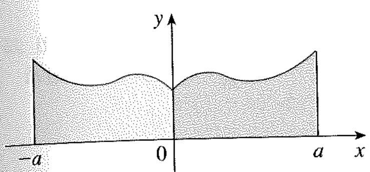
$$\boxed{7} \qquad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

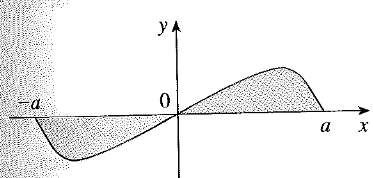
$$-\int_{0}^{-a} f(x)\,dx = -\int_{0}^{a} f(-u)\,(-du) = \int_{0}^{a} f(-u)\,du$$

and so Equation 7 becomes

$$\boxed{8} \qquad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

FIGURE 2

(a) If f is even, then $f(-u) = f(u)$ so Equation 8 gives

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 8 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0$$

Theorem 6 is illustrated by Figure 2. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_{a}^{b} f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. Thus part (b) says the integral is 0 because the areas cancel.

V EXAMPLE 8 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$
$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7}$$

EXAMPLE 9 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

4.5 Exercises

1–6 Evaluate the integral by making the given substitution.

1. $\int \cos 3x\,dx, \quad u = 3x$

2. $\int x(4 + x^2)^{10}\,dx, \quad u = 4 + x^2$

3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\int \dfrac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$

5. $\int \cos^3\theta \sin\theta \, d\theta, \quad u = \cos\theta$

6. $\int \dfrac{\sec^2(1/x)}{x^2}\, dx, \quad u = 1/x$

7–30 Evaluate the indefinite integral.

7. $\int x \sin(x^2)\, dx$

8. $\int x^2 \cos(x^3)\, dx$

9. $\int (3x - 2)^{20}\, dx$

10. $\int (3t + 2)^{2.4}\, dt$

11. $\int (x + 1)\sqrt{2x + x^2}\, dx$

12. $\int \sec^2 2\theta \, d\theta$

13. $\int \sec 3t \tan 3t\, dt$

14. $\int u\sqrt{1 - u^2}\, du$

15. $\int \dfrac{a + bx^2}{\sqrt{3ax + bx^3}}\, dx$

16. $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

17. $\int \sec^2\theta \tan^3\theta \, d\theta$

18. $\int \cos^4\theta \sin\theta \, d\theta$

19. $\int (x^2 + 1)(x^3 + 3x)^4\, dx$

20. $\int \sqrt{x}\sin(1 + x^{3/2})\, dx$

21. $\int \dfrac{\cos x}{\sin^2 x}\, dx$

22. $\int \dfrac{\cos(\pi/x)}{x^2}\, dx$

23. $\int \dfrac{z^2}{\sqrt[3]{1 + z^3}}\, dz$

24. $\int \dfrac{dt}{\cos^2 t\sqrt{1 + \tan t}}$

25. $\int \sqrt{\cot x}\csc^2 x\, dx$

26. $\int \sin t \sec^2(\cos t)\, dt$

27. $\int \sec^3 x \tan x\, dx$

28. $\int x^2\sqrt{2 + x}\, dx$

29. $\int x(2x + 5)^8\, dx$

30. $\int x^3\sqrt{x^2 + 1}\, dx$

31–34 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

31. $\int x(x^2 - 1)^3\, dx$

32. $\int \tan^2\theta \sec^2\theta \, d\theta$

33. $\int \sin^3 x \cos x\, dx$

34. $\int \sin x \cos^4 x\, dx$

35–51 Evaluate the definite integral.

35. $\int_0^1 \cos(\pi t/2)\, dt$

36. $\int_0^1 (3t - 1)^{50}\, dt$

37. $\int_0^1 \sqrt[3]{1 + 7x}\, dx$

38. $\int_0^{\sqrt{\pi}} x\cos(x^2)\, dx$

39. $\int_0^{\pi} \sec^2(t/4)\, dt$

40. $\int_{1/6}^{1/2} \csc \pi t \cot \pi t\, dt$

41. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x)\, dx$

42. $\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$

43. $\int_0^{13} \dfrac{dx}{\sqrt[3]{(1 + 2x)^2}}$

44. $\int_0^a x\sqrt{a^2 - x^2}\, dx$

45. $\int_0^a x\sqrt{x^2 + a^2}\, dx \quad (a > 0)$

46. $\int_{-\pi/3}^{\pi/3} x^4 \sin x\, dx$

47. $\int_1^2 x\sqrt{x - 1}\, dx$

48. $\int_0^4 \dfrac{x}{\sqrt{1 + 2x}}\, dx$

49. $\int_{1/2}^1 \dfrac{\cos(x^{-2})}{x^3}\, dx$

50. $\int_0^{T/2} \sin(2\pi t/T - \alpha)\, dt$

51. $\int_0^1 \dfrac{dx}{(1 + \sqrt{x})^4}$

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52. Verify that $f(x) = \sin\sqrt[3]{x}$ is an odd function and use that fact to show that

$$0 \leqslant \int_{-2}^{3} \sin\sqrt[3]{x}\, dx \leqslant 1$$

53–54 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

53. $y = \sqrt{2x + 1}, \quad 0 \leqslant x \leqslant 1$

54. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

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55. Evaluate $\int_{-2}^{2} (x + 3)\sqrt{4 - x^2}\, dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

56. Evaluate $\int_0^1 x\sqrt{1 - x^4}\, dx$ by making a substitution and interpreting the resulting integral in terms of an area.

57. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

58. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\, dt$, over a 24-hour time period?

59. If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x)\, dx$.

60. If f is continuous and $\int_0^9 f(x)\, dx = 4$, find $\int_0^3 x f(x^2)\, dx$.

61. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

62. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

63. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

64. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

65. If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

66. Use Exercise 65 to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

The following exercises are intended only for those who have already covered Chapter 6.

67–84 Evaluate the integral.

67. $\displaystyle\int \frac{dx}{5-3x}$

68. $\displaystyle\int e^x \sin(e^x)\,dx$

69. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

70. $\displaystyle\int \frac{dx}{ax+b}\quad (a \neq 0)$

71. $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

72. $\displaystyle\int e^{\cos t}\sin t\,dt$

73. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

74. $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx$

75. $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

76. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

77. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

78. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

79. $\displaystyle\int \cot x\,dx$

80. $\displaystyle\int \frac{x}{1+x^4}\,dx$

81. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

82. $\displaystyle\int_0^1 xe^{-x^2}\,dx$

83. $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

84. $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

85. Use Exercise 64 to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

4 Review

Concept Check

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
   (b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
   (c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
2. (a) Write the definition of the definite integral of a continuous function from a to b.
   (b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
   (c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.
3. State both parts of the Fundamental Theorem of Calculus.
4. (a) State the Net Change Theorem.
   (b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?
5. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in meters per second, and acceleration $a(t)$.
   (a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
   (b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
   (c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?
6. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
   (b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?
7. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."
8. State the Substitution Rule. In practice, how do you use it?