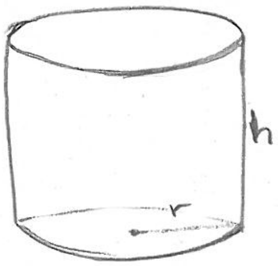
METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY MIDTERM 2	
Code : *MAT 119* Acad.Year : *2013-2014* Semester : *Spring* Date : *03.05.2014* Time : *13:40* Duration : *120 min*	Last Name: Name : Department: Signature : Student No.: Section: 6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (18)	2. (24)	3. (20)	4. (10)	5. (13)	6. (15)	

1. *(18 pts)* Find the largest possible volume of a **closed** cylindrical can which can be made from a metal of $54\pi\ ft^2$.



$\text{Area} = 2\pi r^2 + 2\pi r h = 54\pi$

$\Rightarrow r^2 + rh = 27 \Rightarrow h = \dfrac{27 - r^2}{r}$

$\text{Volume} = \pi r^2 h$

$= \pi r^2 \cdot \dfrac{27 - r^2}{r}$

$V(r) = \pi(27r - r^3), \quad 0 \leq r \leq 3\sqrt{3}$

$V'(r) = \pi(27 - 3r^2) = 0 \Rightarrow r = 3$ (−3 is nonsense)

r	0		3		$3\sqrt{3}$
$V'(r)$		+	0	−	

↑ local max.

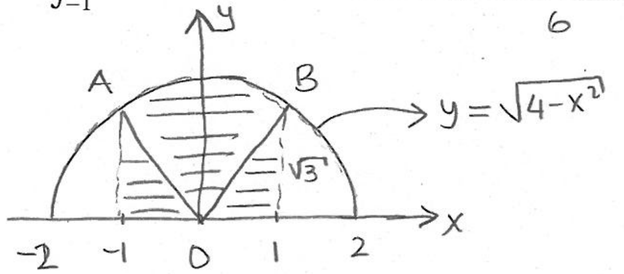
$V(0) = 0$

$V(3) = 54\pi$ — Absolute max

$V(3\sqrt{3}) = 0$

2. *(6×4=24 pts)* Compute the following integrals.

(a) $\int_{-1}^{1} \sqrt{4-x^2}\,dx$ $=$ $\dfrac{\text{Area of the disc}}{6} + 2 \cdot \text{Area of triangles}$



$\Rightarrow \int_{-1}^{1} \sqrt{4-x^2}\,dx = \dfrac{\pi \cdot 2^2}{6} + 2 \cdot \dfrac{1 \cdot \sqrt{3}}{2}$

$= \dfrac{2\pi}{3} + \sqrt{3}$

(b) $\int_{1}^{4} (1 - \frac{1}{\sqrt{x}})(1 + \frac{1}{\sqrt{x}} + \frac{1}{x})\,dx = \int_{1}^{4} \left(1 - \frac{1}{x\sqrt{x}}\right) dx = x + \frac{2}{\sqrt{x}} \Big|_{1}^{4}$

$= \left(4 + \frac{2}{2}\right) - \left(1 + \frac{2}{1}\right)$

$= 2$

(c) $\int_{-1}^{1} (2x^3 - 1)(x^4 - 2x)^6 dx$ $\underset{\substack{u = x^4 - 2x \\ du = (4x^3 - 2)dx}}{=\!=\!=\!=}$ $\int_{3}^{-1} \frac{u^6 du}{2} = \frac{u^7}{14} \Big|_{3}^{-1} = \frac{-1-3^7}{14}$

(d) $\int \sqrt{4 - \sqrt{x}} dx$ $=\!=\!=\!=$ $\int 2(u-4) \cdot \sqrt{u}\, du = 2 \int u\sqrt{u} - 4\sqrt{u}\, du$

$u = 4 - \sqrt{x}$

$x = (4-u)^2$

$dx = 2(4-u) \cdot -du$

$= 2\left(\frac{u^{5/2}}{5/2} - 4 \cdot \frac{u^{3/2}}{3/2}\right) + C$

$= \frac{4}{5}(4-\sqrt{x})^{5/2} - \frac{8}{3}(4-\sqrt{x})^{3/2} + C$

3. *(4×5=20 pts)* Let f and g be two **decreasing continuous** functions on $\mathbb{R}$ with the following properties.

(i) $f(x) = g(x)$ has solutions only at $x = -1$ and $x = 2$.

(ii) $f(-1) = 4$, $f(0) = 2$, $g(0) = 3$ and $g(2) = 0$.

For each part below, write a definite integral which computes the volume of the solid obtained by revolving the region bounded by f and g about the given axis.

(a) about x-axis.

$$\text{Volume} = \int_{-1}^{2} \pi \left(g^2(x) - f^2(x) \right) dx$$

(b) about $y = 10$.

$$\text{Volume} = \int_{-1}^{2} \pi \left[(10 - f(x))^2 - (10 - g(x))^2 \right] dx$$

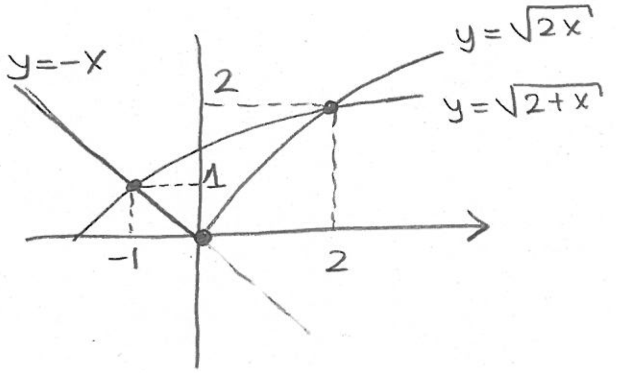
(c) about $x = -3$

$$\text{Volume} = \int_{-1}^{2} 2\pi \, (x - (-3)) \cdot (g(x) - f(x)) \, dx$$

(d) about $x = 5$

$$\text{Volume} = \int_{-1}^{2} 2\pi \, (5 - x) \, (g(x) - f(x)) \, dx$$

4. *(10 pts)* Write a definite integral which computes the area between the curves given by the equations, $y = \sqrt{2+x}$, $y = \sqrt{2x}$ and $y = -x$. (DO NOT EVALUATE)



Intersection points:

$\sqrt{2+x} = \sqrt{2x}$

$\Rightarrow 2+x = 2x \Rightarrow x=2$, $y=2$

$\sqrt{2+x} = -x$

$\Rightarrow 2+x = x^2 \Rightarrow x^2-x-2=0$

~~$x=2$~~; $x=-1$, $y=1$

$\sqrt{2x} = -x$

$\Rightarrow 2x = x^2 \Rightarrow x(x-2)=0$

$x=0$; ~~$x=2$~~, $y=0$

$$\text{Area} = \int_{-1}^{0} \left(\sqrt{2+x} - (-x)\right)dx + \int_{0}^{2} \left(\sqrt{2+x} - \sqrt{2x}\right)dx$$

5. *(7+6=13 pts)* This problem has two unrelated parts.

(a) Find the local maximum/minimum point(s) of the function $g(x) = \int_0^{2x-x^2} \cos(\frac{1}{1+t^2})dt$.

$$g'(x) = \cos\left(\frac{1}{1+(2x-x^2)^2}\right) \cdot (2-2x)$$

$0 < \frac{1}{1+(2x-x^2)^2} \leq 1 \Rightarrow$ only critical point is $x=1$ which makes $g'(1)=0$.

x		1	
$g'(x)$	+		−

local max (at $x=1$)

(b) Show that $2\sqrt{3} < \int_1^3 \sqrt{4x - x^2}dx < 4$. Justify your answer.

Let's find the absolute max & min of $f(x) = \sqrt{4x-x^2}$ on $[1,3]$.

$f'(x) = \frac{4-2x}{2\sqrt{4x-x^2}} \Rightarrow$ critical points; $x=2$ $(f'(2)=0)$

$x=0$; $x=4$ $(f'(0) = \text{undef} = f'(4))$ — not in the domain!

end points: $f(1)=\sqrt{3}$, $f(3)=\sqrt{3}$ → abs. min.

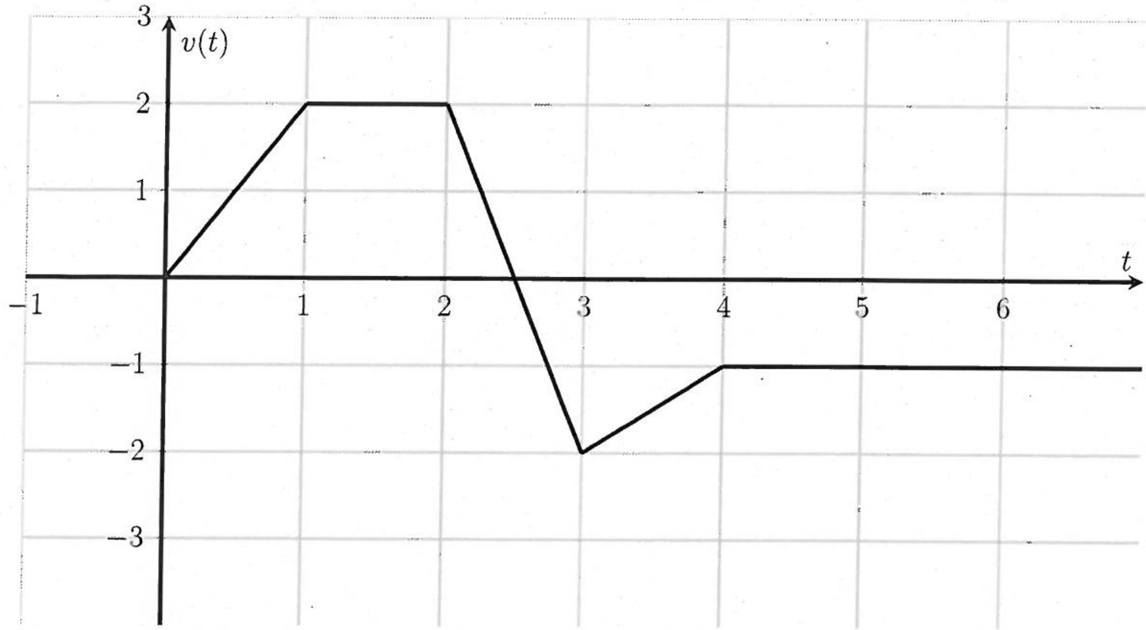
critical point: $f(2)=2$ ← abs. max

$\Rightarrow \sqrt{3} \leq f(x) \leq 2$

$$\Rightarrow \int_1^3 \sqrt{3}\,dx \leq \int_1^3 \sqrt{4x-x^2}\,dx \leq \int_1^3 2\,dx$$

$$\Rightarrow 2\sqrt{3} \leq \int_1^3 \sqrt{4x-x^2}\,dx \leq 4$$

6. *(3×5=15pts)* The velocity function of a particle moving on a straight line is given below.



Answer the following questions by considering the graph above. You don't need to show the details of your work since no partial credits will be given.

(a) Complete the following table considering $s(t)$ as the position function of the particle at time t.

t	0	1	3	5
s(t)	0	1	3	1/2

(b) What is the average velocity on [1,5]?

$$V_{avr} \text{ on } [1,5] = \frac{s(5)-s(1)}{5-1} = \frac{1/2-1}{4} = -\frac{1}{8}$$

(c) What is the total distance travelled on [0,6]?

$$\text{Travelled distance} = \int_0^6 |v(t)|dt = 7.5$$

(d) When does the particle return to the beginning position?

When the area above the x-axis = the area below the x-axis

$\Rightarrow t = 5.5$

(e) What is the function $f(x) = \int_0^x v(t)dt$ equal to for $x \geqslant 5$?

$$f(x) = \int_0^5 v(t)dt + \int_5^x v(t)dt = \frac{1}{2} + \int_5^x -1\,dt = \frac{1}{2} + -t\Big|_5^x = \frac{1}{2} - x + 5$$

$\Rightarrow f(x) = -x + 5.5$