METU - NCC

CALCULUS WITH ANALYTIC GEOMETRY FINAL EXAM		
Code : *MAT 119*	Last Name:	
Acad.Year: *2013-2014*	Name :	Student No.:
Semester : *Spring*	Department:	Section:
Date : *02.06.2014*	Signature :	
Time : *09:00*	6 QUESTIONS ON 6 PAGES	
Duration : *150 min*	TOTAL 105 POINTS	

1. (15)	2. (17)	3. (36)	4. (10)	5. (12)	6. (15)	

1. *(5×3 =15 pts)* Find the following limits. Show and explain your work.

(a) $\lim_{x\to 0} \dfrac{\sec(x)-1}{x^2}$

$$\left(\frac{0}{0}\right) \underset{\text{L'Hospital's Rule}}{=} \lim_{x\to 0} \frac{\sec x\cdot\tan x}{2x} \left(\frac{0}{0}\right) \underset{\text{L'Hospital's Rule}}{=} \lim_{x\to 0} \frac{\sec x\cdot\tan^2 x+\sec^3 x}{2}$$

$$\underset{\text{by Continuity}}{=} \frac{0+1}{2} = \frac{1}{2}$$

(b) $\lim_{x\to 0^+} x^{\sin(x)}$

$$(0^0) = \lim_{x\to 0^+} \left(e^{\ln x}\right)^{\sin x} = \lim_{x\to 0^+} e^{\ln x\cdot\sin x} = e^{\lim_{x\to 0^+} \ln x\cdot\sin x \;(-\infty\cdot 0)}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \left(\frac{-\infty}{\infty}\right)} \underset{\text{L'Hospital's Rule}}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}}$$

$$= e^{\lim_{x\to 0^+} -\overbrace{\left(\frac{\sin x}{x}\right)}^{1 \text{ as } x\to 0^+}\overbrace{\left(\frac{\sin x}{\cos x}\right)}^{0 \text{ as } x\to 0^+}} = e^{1\cdot 0} = 1$$

(c) $\lim_{x\to\infty} \dfrac{\int_1^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,dt}{3x^2}$

$$\int_1^\infty \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,dt = \lim_{x\to\infty}\left[\int_1^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,dt\right] \underset{\substack{u=-\sqrt{t}\\ du=-\frac{1}{2\sqrt{t}}dt}}{=} \lim_{x\to\infty}\left[-\int_{-1}^{-\sqrt{x}} 2\cdot e^u\,du\right] = \lim_{x\to\infty}\left[-2\cdot e^u\Big|_{-1}^{-\sqrt{x}}\right]$$

$$= \lim_{x\to\infty} \underbrace{-2\cdot e^{-\sqrt{x}}}_{\to 0 \text{ as } x\to\infty} - (-2e^{-1}) = \frac{2}{e}$$

$$\lim_{x\to\infty} \frac{\int_1^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,dt}{3x^2} \left(\frac{\frac{2}{e}}{\infty}\right) = 0$$

2. *(6+5+6=17 pts)* This problem has three unrelated parts about derivative of functions.

(a) Write the tangent line equation to the curve $y = \arctan(\pi^{\ln x})$ at $(1, \frac{\pi}{4})$ ...

$$y' = \frac{1}{1+(\pi^{\ln x})^2} \cdot \pi^{\ln x} \cdot \frac{1}{x} \cdot \ln \pi \qquad y'(1) = \frac{1}{1+(\pi^{\ln 1})^2} \cdot \pi^{\ln 1} \cdot \frac{1}{1} \cdot \ln \pi$$

$$= \frac{\ln \pi}{2}$$

$$\left(y - \frac{\pi}{4}\right) = \frac{\ln \pi}{2}(x-1)$$

(b) Compute $\frac{dy}{dx}$ for $y = \frac{\ln(x+e^x) \cdot \arcsin x}{(x+5)^7 \cdot e^{3x}}$

$$\ln y = \ln(\ln(x+e^x)) + \ln(\arcsin x) - 7\ln(x+5) - 3x \cdot \ln(e)$$

Apply derivative:

$$\frac{y'}{y} = \frac{\frac{1+e^x}{x+e^x}}{\ln(x+e^x)} + \frac{\frac{1}{\sqrt{1-x^2}}}{\arcsin x} - 7 \cdot \frac{1}{x+5} - 3$$

$$y' = \frac{\ln(x+e^x) \cdot \arcsin x}{(x+5)^7 \cdot e^{3x}} \cdot \left(\frac{1+e^x}{(x+e^x)\ln(x+e^x)} + \frac{1}{\sqrt{1-x^2} \cdot \arcsin x} - \frac{7}{x+5} - 3 \right)$$

(c) Compute $\frac{dy}{dx}$ for the curve $x^{\sin y} = y^{\cos x}$

$$\ln x^{\sin y} = \ln y^{\cos x}$$

$$\sin y \cdot \ln x = \cos x \cdot \ln y$$

Apply derivative:

$$\cos y \cdot y' \cdot \ln x + \sin y \cdot \frac{1}{x} = -\sin x \cdot \ln y + \cos x \cdot \frac{y'}{y}$$

$$y'\left(\cos y \cdot \ln x - \frac{\cos x}{y}\right) = -\sin x \cdot \ln y - \sin y \cdot \frac{1}{x}$$

$$y' = \frac{-\sin x \cdot \ln y - \frac{\sin y}{x}}{\cos y \cdot \ln x - \frac{\cos x}{y}}$$

3. *(6 × 6 = 36 pts)* Compute the following integrals.

(a) $\displaystyle\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx = \int \frac{u}{\sqrt{1+u^2}}\,du = \frac{1}{2}\int \frac{1}{\sqrt{s}}\,ds$

$u = \ln x \qquad s = 1+u^2$

$du = \frac{1}{x}\,dx \qquad ds = 2u\,du$

$= \frac{1}{2}\cdot\frac{s^{1/2}}{1/2} + C$

$= \sqrt{s} + C$

$= \sqrt{1+u^2} + C = \sqrt{1+(\ln x)^2} + C$

OR: $u = 1+(\ln x)^2 \quad du = 2\ln x\cdot\frac{1}{x}\,dx$

$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{u} + C$

$= \sqrt{1+(\ln x)^2} + C$

(b) $\displaystyle\int \tan^5\theta\sec^3\theta\, d\theta = \int \tan^4\theta\cdot\sec^2\theta\cdot\tan\theta\sec\theta\, d\theta$

$= \int (\sec^2\theta - 1)^2\sec^2\theta\cdot\tan\theta\sec\theta\, d\theta$

$u = \sec\theta$

$du = \sec\theta\cdot\tan\theta\, d\theta$

$= \int (u^2-1)^2\cdot u^2\,du = \int u^6 - 2u^4 - u^2\,du$

$= \frac{u^7}{7} - 2\cdot\frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^7\theta}{7} - 2\cdot\frac{\sec^5\theta}{5} + \frac{\sec^3\theta}{3} + C.$

(c) $\displaystyle\int x\sin^2 x\cos x\, dx = x\cdot\frac{\sin^3 x}{3} - \frac{1}{3}\int \sin^3 x\, dx = \frac{x\sin^3 x}{3} + \frac{\cos x}{3} - \frac{\cos^3 x}{9} + C$

$u = x \qquad du = dx$

$dv = \sin^2 x\cos x\, dx \quad v = \frac{\sin^3 x}{3}$

$\int \sin^3 x\, dx = \int \sin^2 x\,\sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$= -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C$

$= -\cos x + \frac{\cos^3 x}{3} + C$

(d) $\displaystyle\int \frac{1}{x^7 - x}\,dx = \int \frac{x^2}{x^9 - x^3}\,dx \underset{\substack{u = x^3 \\ du = 3x^2dx}}{=} \frac{1}{3}\int \frac{du}{u^3 - u} = \frac{1}{3}\left(\int \frac{A}{u}\,du + \int \frac{B}{u-1}\,du + \int \frac{C}{u+1}\,du \right)$

$$\frac{1}{u(u^2-1)} = \underset{(u^2-1)}{\frac{A}{u}} + \underset{u(u+1)}{\frac{B}{u-1}} + \underset{u(u-1)}{\frac{C}{u+1}} = \frac{(A+B+C)u^2 + (B-C)u - A}{u^3 - u}$$

$$\left.\begin{array}{l} A = -1 \\ B - C = 0 \Rightarrow B = C \\ A+B+C = 0 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2} = C \end{array}\right\} \int \frac{1}{x^7 - x}\,dx = -\frac{1}{3}\ln|u| + \frac{1}{6}\ln|u-1| + \frac{1}{6}\ln|u+1| + C$$

$$= -\frac{1}{3}\ln|x^3| + \frac{1}{6}\ln|x^3 - 1| + \frac{1}{6}\ln|x^3+1| + C$$

(e) $\displaystyle\int_0^2 x^2 \ln x\,dx = \lim_{t\to 0^+}\left[\int_t^2 x^2 \ln x\,dx\right] = \lim_{t\to 0^+}\left[\frac{x^3}{3}\ln x\,\Big|_t^2 - \int_t^2 \frac{x^2}{3}\,dx \right]$

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = x^2 dx \quad v = \frac{x^3}{3}$

$$= \lim_{t\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\Big|_t^2 \right] = \lim_{t\to 0^+}\left[\left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{t^3}{3}\ln t - \frac{t^3}{9}\right)\right] = \underline{\underline{\frac{8}{3}\ln 2 - \frac{8}{9}}}$$

$$\lim_{t\to 0^+} \frac{t^3}{3}\cdot \ln t \ (0\cdot(-\infty)) = \lim_{t\to 0^+} \frac{\ln t}{\frac{3}{t^3}} \left(\frac{-\infty}{\infty}\right) \underset{\text{L'Hospital's Rule}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-9}{t^4}} = \lim_{t\to 0^+} \frac{t^3}{-9} = 0$$

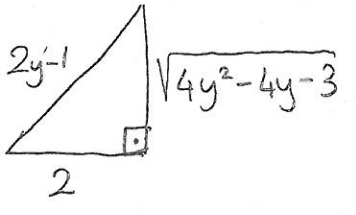
(f) $\displaystyle\int \frac{1}{\sqrt{4y^2 - 4y - 3}}\,dy = \int \frac{dy}{\sqrt{4y^2 - 4y + 1 - 4}} = \int \frac{dy}{\sqrt{(2y-1)^2 - 4}}$

$$= \frac{1}{2}\int \frac{\sec\theta \cdot \tan\theta\,d\theta}{\sqrt{4\sec^2\theta - 4}} = \frac{1}{2}\int \frac{\sec\theta \cdot \tan\theta\,d\theta}{2\tan\theta}$$

$2y - 1 = 2\sec\theta$

$2dy = \sec\theta\tan\theta\,d\theta$

$$= \frac{1}{4}\int \sec\theta\,d\theta = \frac{1}{4}\ln|\sec\theta + \tan\theta| + C$$



$$= \frac{1}{4}\ln\left| \frac{2y-1}{2} + \frac{\sqrt{4y^2-4y-3}}{2} \right| + C$$

4. *(10 pts)* Find the least possible surface area of a **closed** cylindrical can whose volume is equal to $54\pi\ cm^3$.

$\pi r^2 \cdot h = 54\pi\ cm^3$ Minimize $A(r,h) = 2\pi r^2 + 2\pi r h$

$$h = \frac{54}{r^2} \Longrightarrow A(r) = 2\pi r^2 + 2\pi r \cdot \frac{54}{r^2}$$

$$= 2\pi r^2 + \frac{108\pi}{r} \quad (0, +\infty)$$

$$A'(r) = 4\pi r - \frac{108\pi}{r^2} = \frac{4\pi r^3 - 108\pi}{r^2} = 0 \Rightarrow 4\pi r^3 = 108\pi$$

$$r = 3\ cm$$

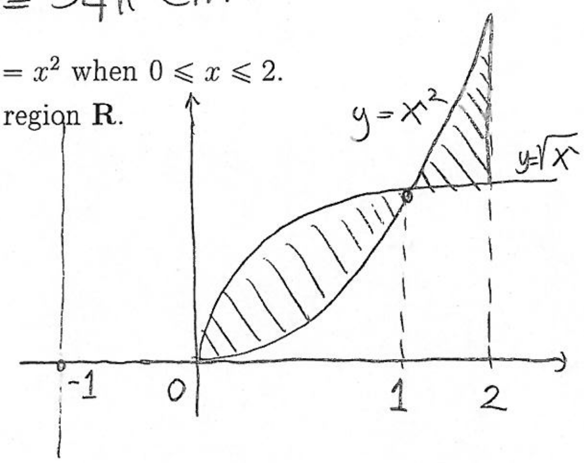
0	3	
	$-$	$+$
	$\searrow$	$\nearrow$

By First Derivative Test for Absolute Extrema, $A(r)$ will have its global minimum at $r = 3$

$$A(3) = 2\pi \cdot 3^2 + \frac{108\pi}{3} = 54\pi\ cm^2$$

5. *(4×3=12 pts)* Let **R** be the region between $y = \sqrt{x}$ and $y = x^2$ when $0 \leqslant x \leqslant 2$.

(a) Write a definite integral which computes the area of the region **R**.



$$\text{Area} = \int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^2 (x^2 - \sqrt{x})\,dx$$

(b) Write a definite integral which computes the volume of the solid obtained by revolving the region **R** about x-axis.(DO NOT EVALUATE)

$$\text{Volume} = \int_0^1 \left[\pi(\sqrt{x})^2 - \pi(x^2)^2\right]dx + \int_1^2 \left[\pi(x^2)^2 - \pi(\sqrt{x})^2\right]dx$$

Cross-Section = Washer

$\text{Area} = \pi r_{out}^2 - \pi r_{in}^2$

(c) write a definite integral which computes the volume of the solid obtained by revolving the region **R** about $x = -1$.(DO NOT EVALUATE)

$$\text{Volume} = \int_0^1 2\pi(x+1)(\sqrt{x} - x^2)\,dx + \int_1^2 2\pi(x+1)\cdot(x^2 - \sqrt{x})\,dx$$

Cylindrical Shell Method

6. *(15 pts)* Given $f(x) = \dfrac{x}{\ln x}$

(a) Find the domain, x-intercepts and y-intercept of $f(x)$.

$\mathrm{Dom}(f) = \mathbb{R}^+ \setminus \{1\}$ No y-intercept

No x-intercept

(b) Find the asymptotes of $f(x)$.

$$\lim_{x\to 1^+} \frac{x}{\ln x} = +\infty \qquad \lim_{x\to 1^-} \frac{x}{\ln x} = -\infty$$

$$\lim_{x\to 0^+} \frac{x}{\ln x} = \lim_{x\to 0^+} x \cdot \frac{1}{\ln x} = 0$$

$$\lim_{x\to\infty} \frac{x}{\ln x} \left(\frac{\infty}{\infty}\right) \underset{\text{L'Hospital's Rule}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = +\infty$$

(c) Find the intervals of increase/decrease and local max/min points of $f(x)$.

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2} = 0 \Rightarrow x = e. \qquad f'(x) \text{ doesn't exist at } x = 1.$$

0	1	e	
	$-$	$-$	$+$
	↘	↘	↗

$$f(e) = \frac{e}{\ln e} = \frac{e}{1}$$

(d) Find the intervals of concavity and inflection points of $f(x)$.

$$f''(x) = \frac{\frac{1}{x}\cdot(\ln x)^2 - (\ln x - 1)\cdot 2\ln x \cdot \frac{1}{x}}{(\ln x)^4} = \frac{\overbrace{\ln x - 2\ln x + 2}^{-\ln x + 2}}{x\cdot(\ln x)^3} = 0 \Rightarrow x = e^2$$

0	1	e^2	
	$-$	$+$	$-$
	Concave Down	Concave up	Concave Down

$$f(e^2) = \frac{e^2}{\ln(e^2)} = \frac{e^2}{2}$$

(e) Sketch the graph of $f(x)$.