M E T U
Northern Cyprus Campus

	Introduction to Differential Equations Final Exam	

Code : *Math 219* Acad.Year: *2010-2011* Semester : *Summer* Date : *16.8.2011* Time : *15:00* Duration : *180 minutes*	Last Name: Name : Department: Signature: Student No: Section: QUESTIONS ON PAGES TOTAL 115 POINTS

1	2	3	4	5	6	7	8		

Show your work! No calculators! Please draw a box around your answers!
Please do not write on your desk!

1. (5+5) This problem has two unrelated parts.

(a) Find the solution to the initial value problem

$$(x^2+1)\frac{dy}{dx}+y=1 \qquad y(0)=2$$

$\frac{dy}{dx}+\frac{1}{x^2+1}\cdot y=\frac{1}{x^2+1}$ (Linear) $\mu(x)=e^{\int\frac{1}{x^2+1}dx}=e^{\tan^{-1}(x)}$

$\frac{d}{dx}\left(e^{\tan^{-1}(x)}\cdot y\right)=e^{\tan^{-1}(x)}\cdot\frac{1}{1+x^2} \Rightarrow y(x)=\frac{\int e^{\tan^{-1}(x)}\cdot\frac{1}{1+x^2}dx}{e^{\tan^{-1}(x)}}$

$u=\tan^{-1}(x)$
$du=\frac{1}{1+x^2}dx$

$\int e^u du=e^u+c=e^{\tan^{-1}(x)}+c$

$y(x)=\frac{e^{\tan^{-1}(x)}+c}{e^{\tan^{-1}(x)}}=1+\frac{c}{e^{\tan^{-1}(x)}}$

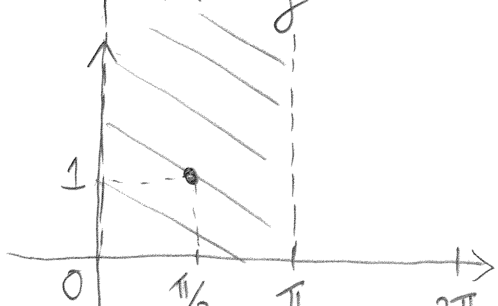
$2=y(0)=1+\frac{c}{e^0}=1+c\Rightarrow c=1$ $\boxed{y(x)=1+\frac{1}{e^{\tan^{-1}(x)}}}$

b) Find the largest rectangle in xy-plane where the following initial value problem has a unique solution.

$$\sin(x)y'-x\ln(y)=0 \quad y(\frac{\pi}{2})=1$$

$y'=\frac{dy}{dx}=\frac{x\cdot\ln y}{\sin x}$ $f(x,y)=\frac{x\cdot\ln y}{\sin x}$ is continuous for $y>0$
$x\neq n\cdot\pi$

$\frac{\partial f}{\partial y}=\frac{x}{\sin x}\cdot\frac{1}{y}$ is continous for $y>0$ and $x\neq n\cdot\pi$



It has a unique solution on
$(0,\pi)\times(0,+\infty)$
by Existence and Uniqueness Theorem.

2.(15) Find the general solution to the non-homogeneous differential equation

$$y'' + 2y' + 2y = te^t$$

Characteristic Polynomial $= r^2+2r+2$ has complex root $-1 \mp i$

Hence $y_h(t) = c_1 e^{-t} \cdot \cos t + c_2 \cdot e^{-t} \sin t$

By using the method of undetermined coefficients, we'll try

$y_p(t) = (At+B) \cdot e^t$

$y_p'(t) = A \cdot e^t + (At+B) \cdot e^t$

$y_p''(t) = A \cdot e^t + A \cdot e^t + (At+B)e^t = 2A \cdot e^t + (At+B)e^t$

Let's put them in the differential equation.

$2A \cdot e^t + \underline{A \cdot t \cdot e^t} + B \cdot e^t + 2Ae^t + \underline{2At \cdot e^t} + 2B \cdot e^t + \underline{2Ate^t} + 2Be^t = t \cdot e^t$

$(4A+5B)e^t + 5A \cdot t \cdot e^t = t \cdot e^t$

$5A = 1 \Rightarrow A = \frac{1}{5}$

$4A + 5B = 0 \qquad B = -\frac{4}{25}$

$$\boxed{y_g(t) = c_1 \cdot e^{-t} \cdot \cos t + c_2 \cdot e^{-t} \cdot \sin t + \left(\frac{1}{5}t - \frac{4}{25}\right)e^t}$$

3.(10+5) A mass of 1 kg is attached to a spring with constant k=1; there is no damping. Let $y(t)$ be the position function of the mass at any time t. At $t = 0$, the mass which is 3 cm below the rest position is released with zero velocity. At the instant $t = \frac{3\pi}{2}$, the mass is struck by a hammer, providing an impulse equal to 2.

(a) Determine the position of the mass at any time t.

We have the spring-mass equation

$$m \cdot y'' + \gamma \cdot y' + k \cdot y = F(t) \qquad \begin{matrix} m = 1 \\ k = 1 \end{matrix} \qquad \begin{matrix} \gamma = 0 \text{ (Undamped} \\ F(t) = 2 \cdot \delta(t - \frac{3\pi}{2}) \end{matrix}$$

$$y'' + y = 2 \cdot \delta(t - \tfrac{3\pi}{2}) \qquad y(0) = 3 \quad y'(0) = 0$$

Apply Laplace transform

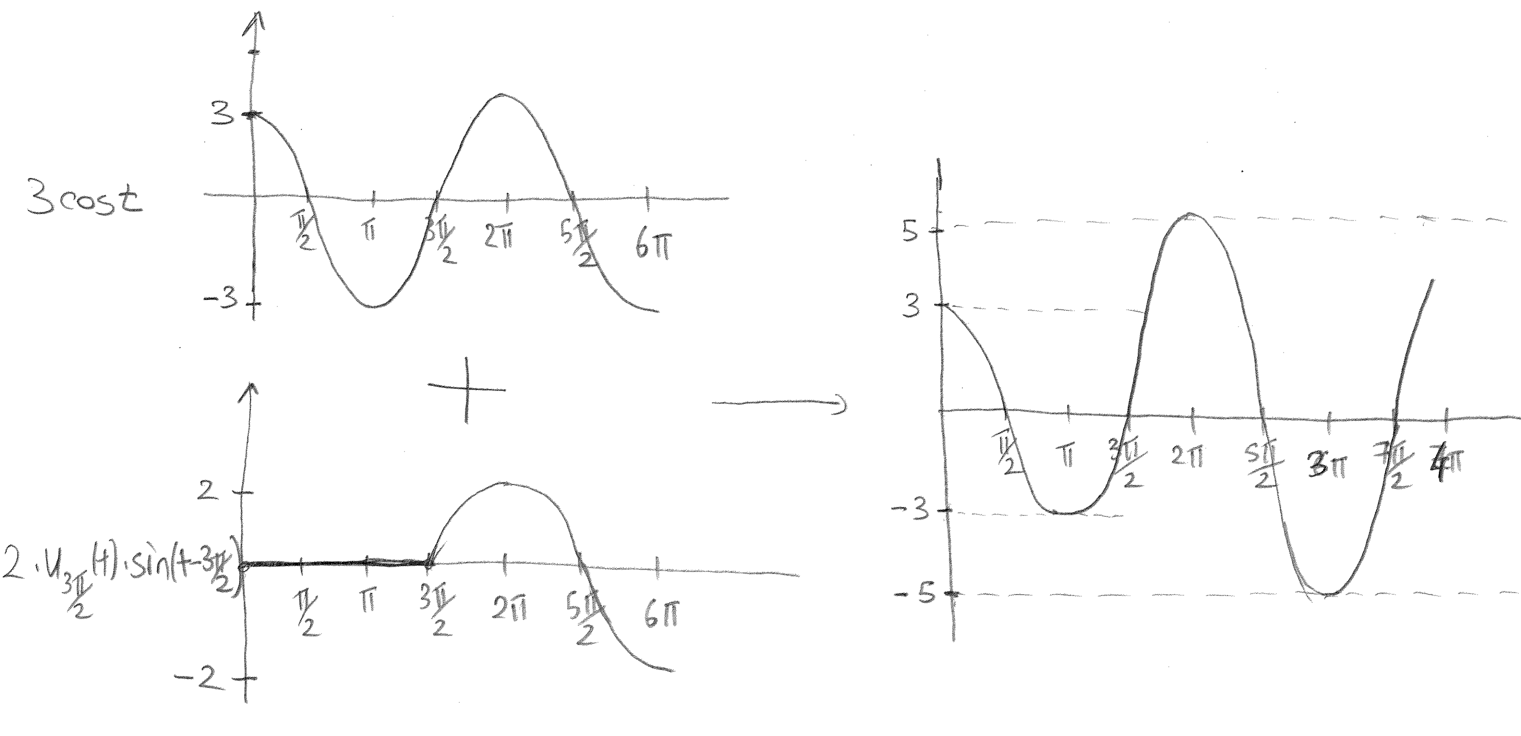
$$s^2 \cdot Y(s) - s \cdot \underbrace{y(0)}_{=3} - \underbrace{y'(0)}_{=0} + Y(s) = 2 \cdot e^{-\frac{3\pi}{2} \cdot s}$$

$$(s^2 + 1)\, Y(s) = 2 \cdot e^{-\frac{3\pi}{2} s} + 3s \Rightarrow Y(s) = 2 \cdot \frac{e^{-\frac{3\pi}{2} s}}{s^2+1} + 3 \cdot \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = 3 \cdot \cos t + 2 \cdot u_{\frac{3\pi}{2}}(t) \cdot \sin(t - \tfrac{3\pi}{2})$$

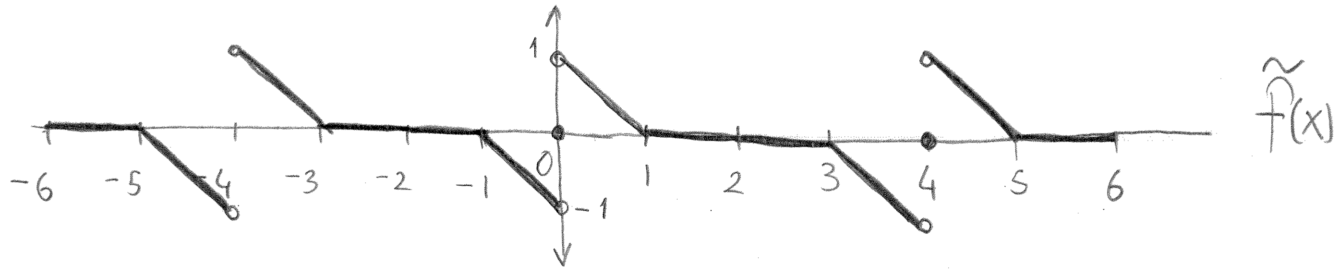
$$= \begin{cases} 3 \cdot \cos t & 0 \leq t \leq 3\pi/2 \\ 5 \cdot \cos t & 3\pi/2 < t \end{cases} \qquad \text{since } \sin(t - \tfrac{3\pi}{2}) = \cos(t$$

(b) Graph your solution.



4.(3+10+2+3+2) Let $f(x) = \begin{cases} 1-x & 0 < x < 1 \\ 0 & 1 \le x \le 2 \end{cases}$

(a) Extend $f(x)$ to all of $\mathbb{R}$ as an odd function with a period 4, and graph your result below.



(b) Calculate the Fourier Series of the function in **(a)**.

Since the extension is odd, we'll have Fourier Sine Series.

$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{\tilde{f}(x)}_{\text{odd}} \cdot \underbrace{\sin\left(\frac{n\pi}{2}\cdot x\right)}_{\text{odd}}}_{\text{even}} dx = 2 \cdot \frac{1}{2}\int_0^2 f(x)\cdot \sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \int_0^1 (1-x)\cdot \sin\left(\frac{n\pi}{2}x\right)dx = -(1-x)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \int_0^1 \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)dx$$

$u = 1-x \quad dv = \sin\left(\frac{n\pi}{2}x\right)dx$

$du = -dx \quad v = \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$

$$= \frac{(2x-2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_0^1 + \frac{4}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}x\right)\Big|_0^1$$

$$= \left[\cancel{\frac{0}{n\pi}\cdot\cos\left(\frac{n\pi}{2}\cdot 1\right)} - \left(\frac{-2}{n\pi}\cdot\cos(0)\right)\right] + \left[\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{4}{n^2\pi^2}\cdot 0}\right]$$

$$= \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{2}{n\pi} + \frac{4}{n^2\pi^2} & n = 4k+1 \\ \frac{2}{n\pi} & n = 4k+2 \\ \frac{2}{n\pi} - \frac{4}{n^2\pi^2} & n = 4k+3 \\ & n = 4k+4 \end{cases}$$

(c) Find the value of the Fourier Series at the following points.

at $x = 2$ $\quad 0$

at $x = -4$ $\quad 0 = \dfrac{\tilde{f}(-4^+) + \tilde{f}(-4^-)}{2}$ by Fourier Convergence Thm.

(d) Find the solution to the heat equation below by using your previous results.

$$u_t = 2u_{xx} \quad u(0,t) = u(2,t) = 0 \quad u(x,0) = f(x)$$

$$U(x,t) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi}{2}x\right) \cdot e^{-\frac{n^2 \cdot \pi^2}{4} \cdot 2t}$$

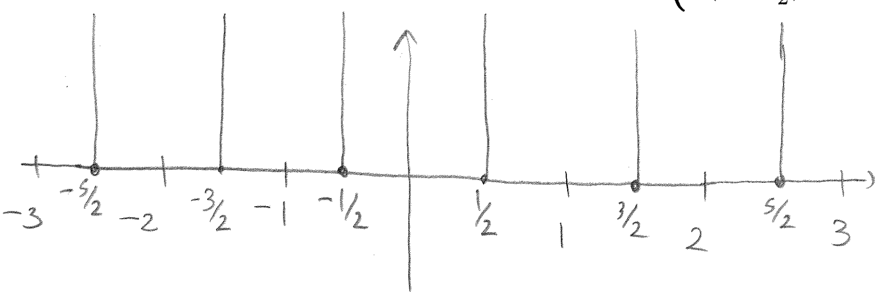
b_n is as in (b).

e) How does $u(x,t)$ behave as $t \longrightarrow \infty$?

$$\lim_{t \to \infty} U(x,t) = 0 \quad \text{since} \quad \lim_{t \to \infty} e^{-\frac{n^2\pi^2}{4} \cdot 2t} = 0$$

5. (8) Find the Fourier Series of the function

$$f(x) = \begin{cases} \delta(x + \frac{1}{2}) & -1 < x < 0 \\ \delta(x - \frac{1}{2}) & 0 < x < 1 \end{cases} \qquad f(x+2) = f(x)$$



$f(x)$ is an even function with period 2.
Hence, we'll have Fourier Cosine Series.

$$a_0 = \frac{1}{1} \cdot \int_{-1}^{1} f(x)dx = 2$$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

$$a_m = \frac{1}{1} \cdot \int_{-1}^{1} \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\cos(n\pi \cdot x)}_{\text{even}}}_{\text{even}} dx = 2 \cdot \int_{0}^{1} \delta(x - \tfrac{1}{2}) \cdot \cos(n\pi \cdot x) dx = 2 \cdot \cos(n\pi \tfrac{1}{2}$$

$$2 \cdot \cos\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^k \cdot 2 & n = 2k \\ 0 & n = 2k+1 \end{cases}$$

$$\boxed{f(x) = \frac{2}{2} + \sum^{\infty} (-1)^k \cdot 2 \cdot \cos(2k \cdot x)}$$

6. (15) Find the solution of the following non-homogeneous system by using the method of variation of parameters.

$$X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad \det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1$$

$\underline{\lambda_1 = 1}$

$$\begin{pmatrix} 2-1 & -1 \\ 3 & -2-1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \xrightarrow{-3R_1+R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$

$$\begin{pmatrix} 2-(-1) & -1 \\ 3 & -2-(-1) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$X_h(t) = c_1 \cdot e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

$\boxed{X_p(t) = \Psi(t) \circ u}$ By Variation of Parameters Method

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \int \Psi(t)^{-1} \cdot g(t)\, dt \quad \text{where} \quad g(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\Psi^{-1}(t) \cdot g(t) = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^t \end{pmatrix} \begin{pmatrix} e^t \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 - e^{-t} \cdot t \\ -e^{2t} + t \cdot e^t \end{pmatrix}$$

$$\boxed{\begin{aligned} u_1(t) &= \frac{1}{2} \int 3 - e^{-t} \cdot t \, dt = \frac{1}{2} \left(3t + e^{-t} \cdot t + e^{-t} \right) \\ u_2(t) &= \frac{1}{2} \int -e^{2t} + t \cdot e^t \, dt = \frac{1}{2} \left(-\frac{e^{2t}}{2} + te^t - e^t \right) \end{aligned}}$$

$$X(t) = X_h(t) + X_p(t) = \Psi(t) \cdot c + \Psi(t) \cdot u$$

7. (10+5+5) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(a) Find the Jordan form of A, i.e. find a matrix T and the Jordan matrix J so that $A = TJT^{-1}$

$$\det(A - \lambda I) = (-1-\lambda)\cdot(2-\lambda)\cdot(2-\lambda) \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 2^{(2)}$$

$\underline{\lambda_1 = -1}$

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{=\xi^{(1)}}$$

$\underline{\lambda_2 = 2}$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{=\xi^{(2)}}$$

To find generalized eigenvector we'll solve $(A - 2I)\eta = \xi^{(2)}$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow{R_1+R_3} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\eta = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ -1/3 \end{pmatrix} \Rightarrow T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}, \quad J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Calculate e^{Jt}

$$e^{Jt} = e^{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \cdot t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & t \cdot e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

(c) Find the general solution to the system $X' = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} X$

$$\underline{\Psi}(t) = T \cdot e^{J \cdot t} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & t \cdot e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} & te^{2t} \\ 0 & 0 & -e^{2t} \\ e^{-t} & 0 & -\frac{e^{2t}}{3} \end{bmatrix}$$

$$X(t) = c_1 \cdot \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t \cdot e^{2t} \\ -e^{2t} \\ -\frac{e^{2t}}{3} \end{pmatrix}$$

8. (2 pts each) Match the following systems of differential equations with their phase portraits given below.

(i) $X' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} X$

$\lambda^2 - 2\lambda + 5$

$\lambda_{1,2} = 1 \mp 2i$

Spiral
Unstable
D

(ii) $X' = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} X$

$\lambda^2 + 4$

$\lambda_{1,2} = \mp 2i$

Center
Stable
F

(iii) $X' = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} X$

$\lambda^2 + 2\lambda + 1$

$\lambda = -1$, repeated

Improper Node

$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector

B

(iv) $X' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} X$

$\lambda^2 + 4\lambda + 3$

$\lambda_1 = -1,\ \lambda_2 = -3$

Node
Asymp. Stable
C

(v) $X' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} X$

$\lambda^2 + 6\lambda + 9$

$\lambda = -3$ repeated

Improper Node

$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvector

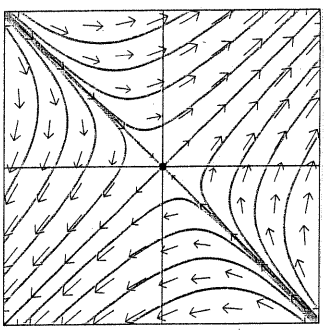
E

(vi) $X' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X$

$\lambda^2 - 2\lambda - 3$

$\lambda_1 = 3 \quad \lambda_2 = -1$

Saddle
Unstable
A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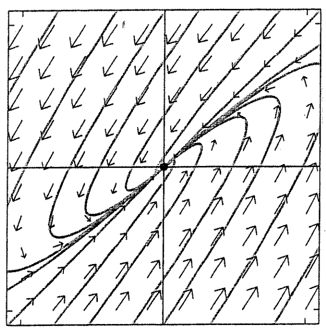
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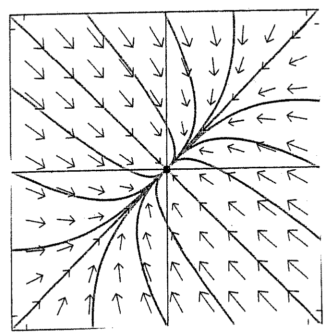
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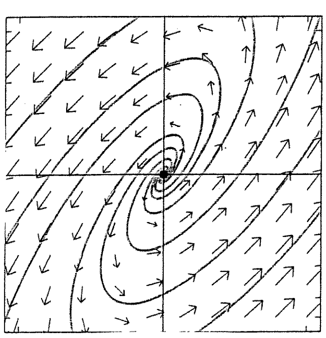
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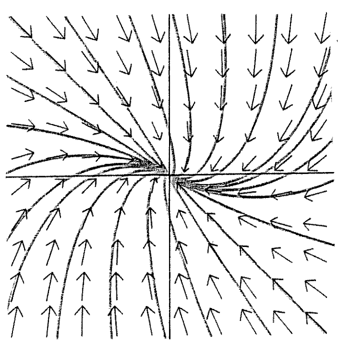
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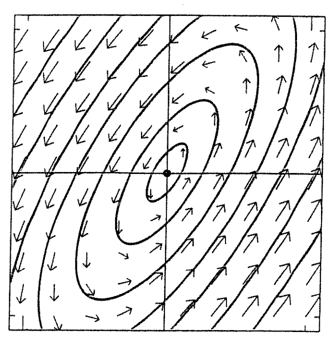
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