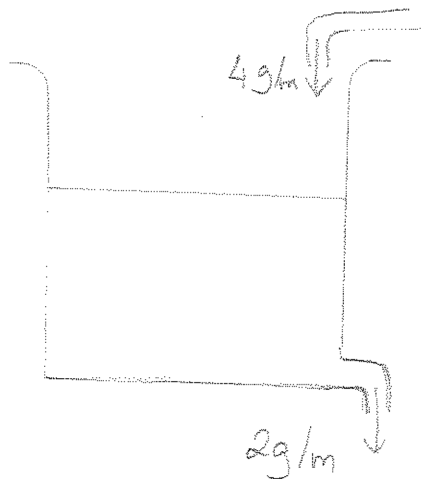
M E T U
Northern Cyprus Campus

Math 219	Differential Equations	Midterm Exam	1.04.2013
Last Name Name : Student No	Dept./Sec.: Time : 17: 40 Duration : 90 *minutes*	Signature	

5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5				

Q1 (20 p.) A tank with capacity of 20 gal originally contains 10 gal of water with 3 lb of salt in solution. Water containing 1 lb salt per gallon enters tank at a rate of 4 gal/min. The mixture flows out of the tank at a rate of 2 gal/ min. Find the amount of salt in the tank when it is on the overflowing level. Finally, **(bonus 5 p.)** show mathematically (to confirm your physical intuition) that the limiting concentration of the salt equals to 1 lb salt per gallon if the tank had infinite capacity.



Let $Q(t)$ be the amount of salt at any time t. So, $Q(0) = 3$ lb.

$$\frac{dQ(t)}{dt} = 4 \cdot 1 - 2 \cdot \frac{Q(t)}{10+(4-2)t}$$

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{5+t} = 4 \Rightarrow \mu(t) = e^{\int \frac{1}{5+t}dt}$$

$$\Rightarrow \mu(t) = 5+t$$

So, $Q(t) = \dfrac{\int 4(5+t)\,dt}{5+t} = \dfrac{2(5+t)^2 + C}{5+t} \Rightarrow Q(t) = 2(5+t) + \dfrac{C}{5+t}$

Since $Q(0) = 3 \Rightarrow 3 = 2(5+0) + \dfrac{C}{5+0} \Rightarrow C = -35$

Our solution becomes, $Q(t) = 2(5+t) - \dfrac{35}{5+t}$

It overflows when $10 + (4-2)t = 20 \Rightarrow t = 5 \Rightarrow Q(5) = 2(5+5) - \frac{35}{5+5} = 6.5$

Bonus: Limiting concentration; $\lim\limits_{t\to\infty} \dfrac{Q(t)}{10+2t} = \lim\limits_{t\to\infty} 1 - \dfrac{35}{2(5+t)^2} = 1$ lb/ga

Q2 (20 p.) Find the fundamental matrix $\Psi(t)$ to solve the following 2×2-linear homogeneous system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 2 & 1 \\ -5 & 4 \end{bmatrix}$.

$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 + 5 \Rightarrow \lambda_1 = 3-2i\,;\ \lambda_2 = 3$

For $\lambda_1 = 3-2i$; $\begin{bmatrix} 2-(3-2i) & 1 \\ -5 & 4-(3-2i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \left.\begin{matrix} (-1+2i)v_1 + v_2 = 0 \\ -5v_1 + (1+2i)v_2 = 0 \end{matrix}\right\} \Rightarrow$

first eigen

For $\lambda_2 = 3+2i$; second eigenvector must be conjugate of the f

so, $\begin{bmatrix} 1 \\ -1-2i \end{bmatrix}$.

Solutions are: $\begin{bmatrix} 1 \\ -1+2i \end{bmatrix} e^{(3-2i)t}$; $\begin{bmatrix} 1 \\ -1-2i \end{bmatrix} e^{(3+2i)t}$

after using Euler's Formula, $(e^{i\theta} = \cos\theta + i\sin\theta)$ our solutions;

$\begin{bmatrix} \cos 2t \\ -\cos 2t + 2\sin 2t \end{bmatrix} e^{3t}$; $\begin{bmatrix} -\sin 2t \\ +2\cos 2t + \sin 2t \end{bmatrix} e^{3t} \Rightarrow \psi(t) = \begin{bmatrix} e^{3t}\cos 2t & -e^{3t}\sin 2t \\ e^{3t}(-\cos 2t + 2\sin 2t) & e^{3t}(+2\cos 2t + s \end{bmatrix}$

Q3 (10 p.) Consider the following IVP $\begin{cases} y' = \dfrac{\tan(t)\sqrt{y}}{1+y} \\ y(0) = -0.5 \end{cases}$. Based on the Existence and Uniqueness Theorem, sketch a largest possible rectangular region about the point $(0,-0.5)$ where the unique solution curve to IVP could exist in (as you know it may not exist globally).

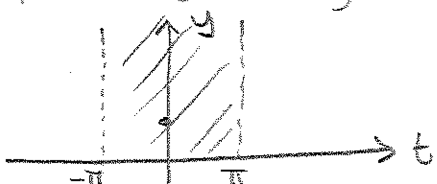
For this non-linear diff eqn; $f(t,y) = \dfrac{\tan(t)\sqrt{y}}{1+y}$, $\dfrac{\partial f}{\partial y}$ both must be

continuous. $\dfrac{\partial f}{\partial y} = \tan(t)\left(\dfrac{\frac{1}{2\sqrt{y}}(1+y) - \sqrt{y}}{(1+y)^2}\right) = \dfrac{\tan(t)\,(1-y)}{2\sqrt{y}\,(1+y)^2}$

Points of discontinuities: $t = \frac{\pi}{2} \mp k\pi$; $y = -1$ and $y \le 0$.

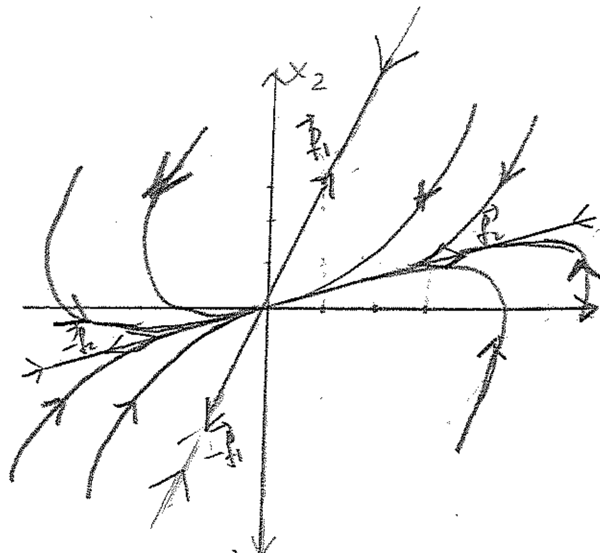
For the given point $(0,+0.5)$, the largest rectangle that we

can choose;

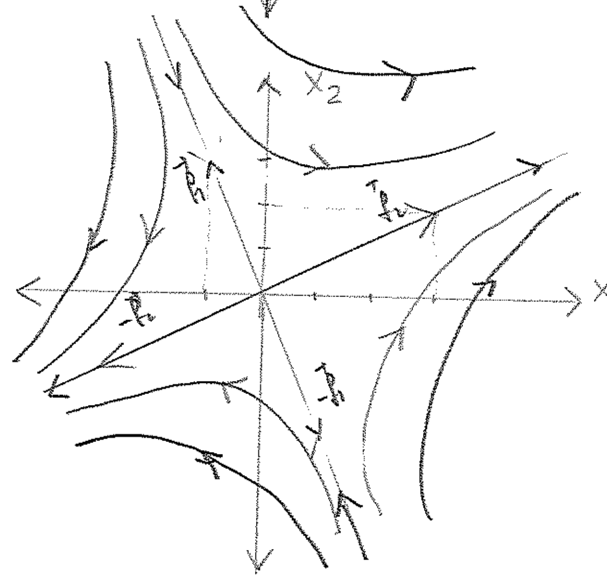


Q4 (30 p.) Sketch the phase portrait of the homogeneous 2×2-linear system $\mathbf{x}' = A\mathbf{x}$ with the set $\sigma(A) = \{\lambda_1, \lambda_2\}$ of its eigenvalues (A is supposed to be a real diagonalizable matrix) and the related independent eigenvectors $\mathbf{f}_1$ and $\mathbf{f}_2$ (or $\mathbf{v}_1$ and $\mathbf{v}_2$), respectively, if

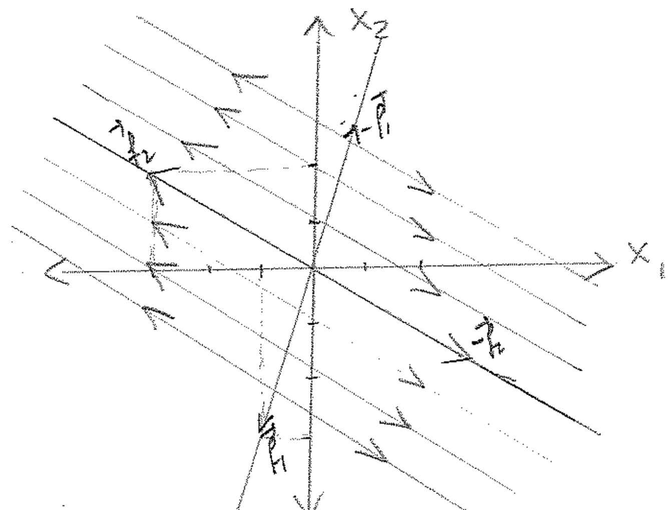
a) $\sigma(A) = \{-5, -3\}$ and $\mathbf{f}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$;



b) $\sigma(A) = \{-2, 5\}$ and $\mathbf{f}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;



c) $\sigma(A) = \{0, 7\}$ and $\mathbf{f}_1 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$;



Q5 (20 p.) Find the fundamental matrix $\Phi(t)$ to solve the following 3×3-linear homogeneous system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ -4 & 4 & -5 \end{bmatrix}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 0 \\ -4 & 4 & -5-\lambda \end{bmatrix} = (2-\lambda)\big((1-\lambda)(-5-\lambda) - 2\cdot -4\big) = (2-\lambda)(\lambda^2+4\lambda \ldots$$

$$= (2-\lambda)(\lambda+1)(\lambda+ \ldots$$

$$\Rightarrow \lambda_1 = -3;\ \lambda_2 = -1;\ \lambda \ldots$$

For $\lambda_1 = -3$:

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 0 \\ -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} 4v_1 + v_2 + 2v_3 &= 0 \\ 5v_2 &= 0 \\ -4v_1 + 4v_2 - 2v_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} v_2 &= 0 \\ v_3 &= -2v_1 \end{aligned} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

For $\lambda_2 = -1$:

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 0 \\ -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} 2v_1 + v_2 + 2v_3 &= 0 \\ 3v_2 &= 0 \\ -4v_1 + 4v_2 - 4v_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} v_2 &= 0 \\ v_3 &= -v_1 \end{aligned} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda_3 = 2$:

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ -4 & 4 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} -v_1 + v_2 + 2v_3 &= 0 \\ 0 &= 0 \\ -4v_1 + 4v_2 - 7v_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} v_3 &= 0 \\ v_2 &= v_1 \end{aligned} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So, $T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$ and its inverse; $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_1 + R_3 \to R_3}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_2 + R_1 + R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & - \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} -1 & 1 & -1 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{matrix}}_{T^{-1}}$$

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} & e^{2t} \\ 0 & 0 & e^{2t} \\ -2e^{-3t} & -e^{-t} & 0 \end{bmatrix} \Rightarrow \Phi(t) = \Psi(t)\,T^{-1} = \begin{bmatrix} -e^{-3t} + 2e^{-t} & e^{-3t} - 2e^{-t} + e^{2t} & -e^{-3t} + e^{-t} \\ 0 & e^{2t} & 0 \\ 2e^{-3t} - 2e^{-t} & -2e^{-3t} + 2e^{-t} & 2e^{-3t} - e^{-t} \end{bmatrix}$$