# METU - NCC

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| | **Introduction to Differential Equations**<br>**Midterm 1** | |

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| Code : *MAT 219*<br>Acad.Year: *2012-2013*<br>Semester : *Fall*<br>Date : *18.10.2012*<br>Time : *17:40*<br>Duration : *120 minutes* | Last Name:<br>Name : KEY<br>Department:<br>Signature:<br>Student No.:<br>Section:<br>5 QUESTIONS ON 5 PAGES<br>TOTAL 100 POINTS |

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1. *(20 pts)* Find the value of $b$ that makes the differential equation exact, and solve the initial value problem below. (You can leave your answer in implicit form.)

$$\underbrace{(xy^2 + bx^2y)}_{M}dx + \underbrace{(x+y)x^2}_{N}dy = 0, \qquad y(2) = 3$$

Using the Component Test, we have

$(M_y = 2xy + bx^2 = N_x = 3x^2 + 2xy) \Leftrightarrow b = 3.$

In this case there is a potential function $\Psi(x,y)$:

$$\begin{cases} \Psi_x = xy^2 + 3x^2y \\ \Psi_y = x^3 + x^2y \end{cases} \begin{array}{l} \Rightarrow \Psi = \frac{1}{2}x^2y^2 + x^3y + C(y) \\ \Downarrow \\ \Rightarrow x^3 + x^2y = x^2y + x^3 + C'(y) \end{array}$$

Hence $C'(y) = C$ and $\frac{1}{2}x^2y^2 + x^3y = C$ is the general solution in the implicit form.

IVP: $C = \Psi(2,3) = \frac{1}{2}4\cdot 9 + 8\cdot 3 = 42$,

$\frac{1}{2}x^2y^2 + x^3y = 42$ is the solution to IVP.

**2.** *(20 pts)* Consider the initial value problem

$$ty' + y = g(t), \quad y(-\pi/2) = 0$$

Assume that the function $g(t)$ is continuous for all values of $t$.

(a) Find the largest interval for which a solution of this initial value problem is sure to exist. Explain your answer.

The canonical form: $y' + \frac{1}{t}y = \frac{g(t)}{t}$. The functions $p(t) = \frac{1}{t}$, $q(t) = \frac{g(t)}{t}$ are continuous on $\mathbb{R} \setminus \{0\}$, and the initial point is negative. By Existence and Uni Th., $I = (-\infty, 0)$.

(b) Solve the associated homogenous equation $ty' + y = 0$.

$t\frac{dy}{dt} = -y \Rightarrow \frac{dy}{y} = -\frac{dt}{t}$ $(y \neq 0) \Rightarrow \ln|y| = -\ln|Ct|$, C

$\Rightarrow |y| = \frac{C}{|t|}, C > 0 \Rightarrow y = \frac{C}{t}, C \neq 0$. But $y = 0$ i

a solution either $\Rightarrow y = \frac{C}{t}$ is the general solutio to the associated homg. dif equation.

(c) Let $g(t) = 2\sin(2t)$. Find a particular solution of the equation $ty' + y = g(t)$ using variation of parameters.

Put $Y(t) = \frac{C(t)}{t}$. Then $Y'(t) = \frac{C'(t)}{t} - \frac{C(t)}{t^2} \Rightarrow$

$\Rightarrow tY'(t) + Y(t) = t\left(\frac{C'(t)}{t} - \frac{C(t)}{t^2}\right) + \frac{C(t)}{t} = 2\sin(2t)$

$\Rightarrow C'(t) = 2\sin(2t) \Rightarrow C(t) = -\cos(2t) + C \Rightarrow$

$Y(t) = -\frac{\cos(2t)}{t}$ is a special sol. to nonhomg.

(d) Find the solution of the initial value problem $ty' + y = 2\sin(2t), y(-\pi/2) = 0$ using the results above.

$y(t) = \frac{C}{t} - \frac{\cos(2t)}{t}$ is the general solution to the nonh. eq. Then $0 = -\frac{2C}{\pi} + \frac{2}{\pi}\cos\left(-2\cdot\frac{\pi}{2}\right) = -\frac{2}{\pi}(C+1) \Rightarrow$

$\Rightarrow C = -1$. Thus

$y(t) = -\frac{1 + \cos(2t)}{t}$ is the solution to IVP

3. *(20 pts)* A tank with a capacity of $10\ell$ originally contains $5\ell$ of water with $2g$ of salt in solution. Water containing $\frac{1}{3}g$ of salt per liter (concentration) is entering at a rate of $4\ell/min$, and the mixture is allowed to flow out of the tank at a rate of $1\ell/min$. Find the amount of salt in the tank at any time, and predict the concentration of salt in the tank at the point of overflowing. **Bonus:** Find the theoretical limiting concentration if the tank had infinite capacity, and compare the result with your physical intuition.

$r_1 = 4$, $r_2 = 1 \Rightarrow r = r_1 - r_2 = 4 - 1 = 3$. The amount of water $V(t)$ in the tank is increasing versus time $t$ as $V(t) = 5 + 3t$. Overflowing will take place when $t = t^*$, $V(t^*) = 5 + 3t^* = 10 \Rightarrow$ $\Rightarrow t^* = 5/3$. Moreover, the concentration of salt that flows out of the tank at time $t$ is $\dfrac{Q(t)}{V(t)} = \dfrac{Q(t)}{5+3t}$, where $Q(t)$ is the amount of salt at time $t$. We have IVP

$$\begin{cases} Q'(t) = \frac{1}{3} \cdot 4 - \dfrac{Q(t)}{5+3t} \\ Q(0) = 2 \end{cases} \Rightarrow \mu(t) = e^{\int \frac{dt}{5+3t}} = (5+3t)^{1/3}, \quad t \le 5/3.$$

We have $(\mu \cdot Q)' = \frac{4}{3}(5+3t)^{1/3} \Rightarrow \mu \cdot Q = \frac{1}{3}(5+3t)^{4/3} + C$

$\Rightarrow Q(t) = \dfrac{C}{(5+3t)^{1/3}} + \frac{1}{3}(5+3t)$, $2 = Q(0) = \frac{5}{3} + \dfrac{C}{5^{1/3}} \Rightarrow$

$\Rightarrow C = \dfrac{5^{1/3}}{3} \Rightarrow Q(t) = \dfrac{5}{3} + t + \dfrac{5^{1/3}}{3(5+3t)^{1/3}}$

If $t = t^* = 5/3$ then $Q(5/3) = \dfrac{10}{3} + \dfrac{5^{1/3}}{3 \cdot 10^{1/3}} = \dfrac{10}{3} + \dfrac{1}{3 \cdot 2^{1/3}} \Rightarrow$

$\Rightarrow \dfrac{Q(5/3)}{10} = \dfrac{1}{3} + \dfrac{1}{30 \cdot 2^{1/3}}$ – the concentration of salt at $t = t^*$.

Bonus: $\lim\limits_{t\to\infty} \dfrac{Q(t)}{5+3t} = \lim\limits_{t\to\infty} \left( \dfrac{1}{3} + \dfrac{5^{1/3}}{3(5+3t)^{4/3}} \right) = 1/3$

4. *(20 pts)* Solve the equation

$$\frac{dy}{dx} = \frac{x+3y}{x-y} = \frac{1+3\,y/x}{1-y/x}$$

It is a homogeneous differential equation. Put $\frac{y}{x} = v$.

Then $y' = v + xv' = \frac{1+3v}{1-v} \Rightarrow xv' = \frac{1+3v-v+v^2}{1-v} = \frac{(1+v)^2}{1-v}$

$\Rightarrow \frac{(1-v)\,dv}{(1+v)^2} = \frac{dx}{x} \quad (v \neq -1)$

It follows that $\int \frac{(1-v)\,dv}{(1+v)^2} = \ln|Cx|,\ C>0$

But $\int \frac{(1-v)\,dv}{(1+v)^2} = \left| \begin{matrix} u = 1+v \\ du = dv \\ v = u-1 \end{matrix} \right| = \int \frac{2-u}{u^2}\,du = -\frac{2}{u} - \ln|u| + C$

$= -\frac{2}{1+v} - \ln|1+v| + C$

Whence $\frac{-2}{1+v} = \ln|Cx(1+v)| \Rightarrow \frac{-2x}{x+y} = \ln|Cx(y+x)|, C$

But $y = -x$ is a solution: $y' = -1 = \frac{x+3(-x)}{x-(-x)}$, that we lost in the separation process.

Thus

$$\begin{cases} \frac{-2x}{y+x} = \ln|C(y+x)|,\ C>0, \\ y = -x \end{cases}$$

is then general solution

5. *(7+7+6=20 pts)* (a) Let $a, \omega$ be arbitrary real numbers. Show that the matrix $A$ below is invertible for all values of $t$, and find $A^{-1}$:

$$A = \begin{bmatrix} e^{at}\cos\omega t & -e^{at}\sin\omega t \\ e^{at}\sin\omega t & e^{at}\cos\omega t \end{bmatrix} = A(t)$$

$\det(A(t)) = e^{2at}(\cos^2(\omega t) + \sin^2(\omega t)) = e^{2at} \neq 0, \ \forall t \in$

Then $A(t)^{-1} = e^{-2at}\begin{bmatrix} e^{at}\cos(\omega t) & e^{at}\sin(\omega t) \\ -e^{at}\sin(\omega t) & e^{at}\cos(\omega t) \end{bmatrix} =$

$= \begin{bmatrix} e^{-at}\cos(\omega t) & e^{-at}\sin(\omega t) \\ -e^{-at}\sin(\omega t) & e^{at}\cos(\omega t) \end{bmatrix} = A(-t).$

(b) Let $B$ and $C$ be $2 \times 2$ matrices. Is it always true that $B^2 - C^2 = (B-C)(B+C)$ ? Prove or give a counterexample.

Note that $(B-C)(B+C) = B^2 + BC - CB - C^2 \neq B^2 - C^2$ unless $BC = CB$. But $BC \neq CB$ for the following matrices: $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$;

$BC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq CB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Find all complex values of $z$ that make $det(D)$ zero.

$$D = \begin{bmatrix} i & 1+i & z \\ 5i & 5+5i & 3-i \\ 1 & 0 & 0 \end{bmatrix} = D(z)$$

$\det(D(z)) = (1+i)(3-i) - 5(1+i)z = 4+2i - 5(1+i)z.$

Hence $\det(D(z)) = 0 \Leftrightarrow z = \frac{2(2+i)}{5(1+i)} = \frac{2}{5}\frac{(2+i)(1-i)}{(1+i)(1-i)} =$

$= \frac{2}{5}\,\frac{3-i}{2} = \frac{1}{5}(3-i).$