METU
Northern Cyprus Campus

Math 219	Differential Equations	Midterm Exam	24.04.2011
Last Name: Name: Student No: KEY	Dept./Sec.: Time : 14: 00 Duration : 110 *minutes*	Signature	
6 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS

1	2	3	4	5	6			

Question 1 (20 p.) A thermometer which has been showing $10°C$ inside of a house is placed outside, where the temperature varies as the function $\sin(t)$, where t indicates time. Find the temperature shown by the thermometer at any time t. (*Comment: Certainly, there are different kind of thermometers which can be characterized in terms of positive constants* $k > 0$). Finally, (**bonus 10 p.**) show that if $k = 1$ then the temperature ~~after~~ at 1min will be less that $6.5°C$.

Let $u(t)$ be the temperature function of the thermometer. Due to Newton's Law of cooling, we have $u'(t) = -k(u(t) - \sin(t))$, $k > 0$ and $u(0) = 10$. We deal with linear dif. eq. $u' + ku = k\sin(t)$, $p(t) = k$, $q(t) = k\sin(t)$. Integrating factor $\mu(x) = e^{kt} \Rightarrow (e^{kt}u)' = k e^{kt}\sin(t) \Rightarrow e^{kt}u = k\int e^{kt}\sin(t)\,dt + C$. Using Intg. by Parts, we derive that

$$\int e^{kt}\sin(t)\,dt = -\frac{1}{1+k^2}e^{kt}\cos(t) + \frac{k}{1+k^2}e^{kt}\sin(t).$$

Hence $u(t) = Ce^{-kt} - \frac{k}{1+k^2}\cos(t) + \frac{k^2}{1+k^2}\sin(t)$

(optionally, you could use Method of Undeter. Coefficients)

But $10 = u(0) = C - \frac{k}{1+k^2} \Rightarrow C = 10 + \frac{k}{1+k^2} \Rightarrow$

$$u(t) = \left(10 + \frac{k}{1+k^2}\right)e^{-kt} - \frac{k}{1+k^2}\cos(t) + \frac{k^2}{1+k^2}\sin(t).$$

Finally, assume that $k = 1$. Then

$$|u(1)| \le \frac{21}{2\cdot e} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{21}{2\cdot e} < 1 + \frac{21}{4} = \frac{25}{4} < 6.5$$

Question 2 (15 p.) Find an implicitly defined general solution to the following nonlinear differential equation $y' = \frac{x^2+3y^2}{2xy}$. It is a homogeneous dif. eq.

Put $y = xv(x) \Rightarrow y' = v + xv' = \frac{1}{2v} + \frac{3v}{2} \Rightarrow$

$\Rightarrow xv' = \frac{1}{2v} + \frac{v}{2} = \frac{1+v^2}{2v} \Rightarrow \frac{2v\,dv}{1+v^2} = \frac{dx}{x} \Rightarrow$

$\int \frac{2v\,dv}{1+v^2} = \int \frac{dx}{x} + C \Rightarrow \ln(1+v^2) = \ln|Cx| \Rightarrow$

$1+v^2 = |Cx| \Rightarrow \left(\frac{y}{x}\right)^2 = Cx-1$ or $y^2 = (Cx-1)x^2$.

Question 3 (20 p.) Find the inverse Laplace transform of the following function $G(s) = \frac{e^{-31s}}{(s+1)(s^2+s+1)}$. (*Comment: Do not use the convolution theorem*). First note that $g(t) = \mathcal{L}^{-1}\{G(s)\} = u_{31}(t) f(t-31)$, where $f(t) = \mathcal{L}^{-1}\{F(s)\}$,

$F(s) = \frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$. But

$As^2 + As + A + Bs^2 + Cs + Bs + C = (A+B)s^2 + (A+B+C)s + A+C = 1$

$\Rightarrow A = 1,\ B = -1,\ C = 0$. It follows that

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-t} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\}$.

Further, $\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{4})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{s+1/4}{(s+1/4)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} -$

$-\frac{1}{4} \frac{\frac{\sqrt{3}}{2}}{(s+1/4)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \frac{2}{\sqrt{3}}$. Hence

$f(t) = e^{-t} - e^{-t/4}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2\sqrt{3}} e^{-t/4} \sin\left(\frac{\sqrt{3}}{2}t\right)$ and

$g(t) = u_{31}(t)\left[e^{-(t-31)} - e^{-\frac{t-31}{4}}\cos\left(\frac{\sqrt{3}}{2}(t-31)\right) + \frac{1}{2\sqrt{3}} e^{-\frac{t-31}{4}} \sin\left(\frac{\sqrt{3}}{2}(t-31)\right)\right]$

Question 4 (20 p.) Consider the following nonhomogeneous linear differential equation $ty'' - y' + 4t^3 y = 4t^3$, $t > 0$. The function $y_1(t) = \cos(t^2)$ is a solution to the relevant homogeneous differential equation. Based on $y_1(t)$ find the general solution to the given differential equation.

Reduction of order: $y = \cos(t^2) v(t) \Rightarrow y' = -2t\sin(t^2) v$
$+ \cos(t^2) v' \Rightarrow y'' = -2\sin(t^2) v - 4t^2\cos(t^2) v - 4t\sin(t^2) v' + \cos(t^2) v''$.

$ty'' - y' + 4t^3 y = -2t\sin(t^2) v - 4t^3\cos(t^2) v - 4t^2\sin(t^2) v' +$
$+ t\cos(t^2) v'' + 2t\sin(t^2) v - \cos(t^2) v' + 4t^3\cos(t^2) v =$
$= t\cos(t^2) v'' - (\cos(t^2) + 4t^2\sin(t^2)) v' = 0$ or

$\frac{dv'}{dt} = \left[\frac{1}{t} + 4t\tan(t^2)\right] v' \Rightarrow \ln|v'| = \ln(t) +$
$+ 2\int 2t\tan(t^2)\,dt + C = \ln\left|\frac{ct}{\cos^2(t^2)}\right| = \ln|ct\sec^2(t^2)|$

$\Rightarrow v' = Ct\sec^2(t^2) + k \Rightarrow v = C\tan(t^2) + k \Rightarrow v = \tan(t^2)$
$\Rightarrow y_2(t) = \sin(t^2)$. The canonical form: $y'' - \frac{1}{t}y' + 4t^2 y = 4t^2;\ t > 0$
Variation of Parameters: $Y(t) = u_1\cos(t^2) + u_2\sin(t^2)$

$$\begin{bmatrix} \cos(t^2) & \sin(t^2) \\ -2t\sin(t^2) & 2t\cos(t^2) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 4t^2 \end{bmatrix} \Rightarrow$$

$$u_1' = \frac{1}{2t}\begin{vmatrix} 0 & \sin(t^2) \\ 4t^2 & 2t\cos(t^2) \end{vmatrix} = -2t\sin(t^2) \Rightarrow u_1 = \cos(t^2)$$

$$u_2' = \frac{1}{2t}\begin{vmatrix} \cos(t^2) & 0 \\ -2t\sin(t^2) & 4t^2 \end{vmatrix} = 2t\cos(t^2) \Rightarrow u_1 = \sin(t^2)$$

Hence $Y(t) = -\cos^2(t^2) + \sin^2(t^2) = -\cos(2t^2)$ and

$y(t) = C_1\cos(t^2) + C_2\sin(t^2) - \cos(2t^2)$ - general solution to the nonhomg. dif. eq.

Question 5 (15 p.) Consider a forced undamped mechanical vibration with the unit mass and spring constant 4, whose external force is given by the function $3\cos(2t)$. Write down the differental equation describing the motion $u(t)$ of the system with the intial conditions $u(0) = u'(0) = 0$. Then solve the IVP indicating to the link between duplication and resonance phenomenon. Sketch the graph of the solution.

Since $m=1$, $k=4$, $\gamma=0$, we have the following IVP

$$\begin{cases} u'' + 4u = 3\cos(2t) \\ u(0)=u'(0)=0 \end{cases}$$

Moreover, $\omega_0 = 2 = \omega$ – resonance case. Obviously, $u_h(t) = C_1\cos(2t) + C_2\sin(2t)$.

To find out a special solution, we set

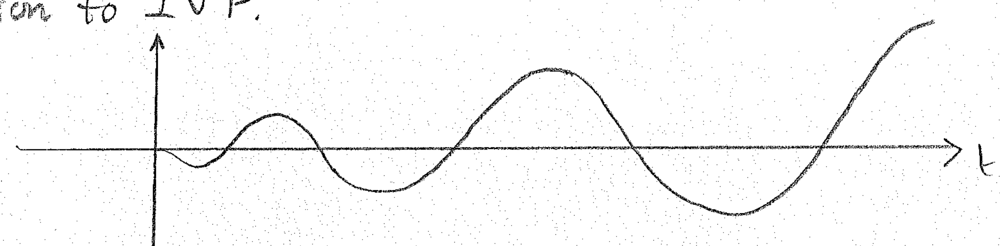
$Y(t) = t(A\cos(2t) + B\sin(2t))$ – duplication, which sustains resonance. $Y'(t) = A\cos(2t) + B\sin(2t) + 2t(-A\sin(2t) + B\cos(2t))$,

$Y''(t) = 4B\cos(2t) - 4A\sin(2t) + 4t(-A\cos(2t) - B\sin(2t)) \Rightarrow$

$Y'' + 4Y = 4B\cos(2t) - 4A\sin(2t) = 3\cos(2t) \Rightarrow B = \frac{3}{4}, A = 0.$

So, $u(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{3}{4}t\sin(2t)$ – general solution

Since $0 = u(0) = u'(0)$, it follows that $u(t) = \frac{3}{4}t\sin(2t)$ is the solution to IVP.



Question 6 (10 p.) Find the general solution to the higher order linear homogeneous differential equation $y''' - y = 0$.

The characteristic eq. $r^3 - 1 = 0 \Rightarrow r = 1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$.

But $e^{i\frac{2\pi}{3}}$ and $e^{i\frac{4\pi}{3}}$ are complex conjugate pairs.

So, $r = 1^{①}, e^{i\frac{2\pi}{3}}$ ① Since $e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$, it follows that

$$y(t) = C_1e^t + C_2e^{\cos\left(\frac{2\pi}{3}\right)t}\cos\left(\sin\left(\frac{2\pi}{3}\right)t\right) + C_3e^{\cos\left(\frac{2\pi}{3}\right)t}\sin\left(\sin\left(\frac{2\pi}{3}\right)t\right)$$

– general solution to the dif. eq.