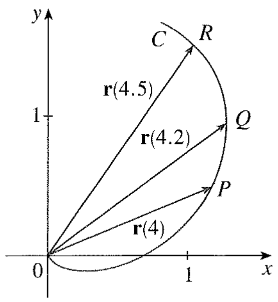
13.2 Exercises

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.
(a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.
(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.
(d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t \rangle$, $0 \leqslant t \leqslant 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.
(b) Draw the vector $\mathbf{r}'(1)$ starting at (1, 1), and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3–8
(a) Sketch the plane curve with the given vector equation.
(b) Find $\mathbf{r}'(t)$.
(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle, \quad t = -1$

4. $\mathbf{r}(t) = \langle t^2, t^3 \rangle, \quad t = 1$

5. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}, \quad t = \pi/4$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}, \quad t = 0$

7. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}, \quad t = 0$

8. $\mathbf{r}(t) = (1 + \cos t)\,\mathbf{i} + (2 + \sin t)\,\mathbf{j}, \quad t = \pi/6$

9–16 Find the derivative of the vector function.

9. $\mathbf{r}(t) = \langle t \sin t, t^2, t \cos 2t \rangle$

10. $\mathbf{r}(t) = \langle \tan t, \sec t, 1/t^2 \rangle$

11. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t}\,\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k}$

13. $\mathbf{r}(t) = e^{t^2}\,\mathbf{i} - \mathbf{j} + \ln(1 + 3t)\,\mathbf{k}$

14. $\mathbf{r}(t) = at \cos 3t\,\mathbf{i} + b \sin^3 t\,\mathbf{j} + c \cos^3 t\,\mathbf{k}$

15. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c}$

16. $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c})$

17–20 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

17. $\mathbf{r}(t) = \langle te^{-t}, 2 \arctan t, 2e^t \rangle, \quad t = 0$

18. $\mathbf{r}(t) = \langle t^3 + 3t, t^2 + 1, 3t + 4 \rangle, \quad t = 1$

19. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2 \sin 2t\,\mathbf{k}, \quad t = 0$

20. $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}, \quad t = \pi/4$

21. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

22. If $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = 1 + 2\sqrt{t}, \quad y = t^3 - t, \quad z = t^3 + t; \quad (3, 0, 2)$

24. $x = e^t, \quad y = te^t, \quad z = te^{t^2}; \quad (1, 0, 0)$

25. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

26. $x = \sqrt{t^2 + 3}, \quad y = \ln(t^2 + 3), \quad z = t; \quad (2, \ln 4, 1)$

27. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point (3, 4, 2).

28. Find the point on the curve $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, e^t \rangle$, $0 \leqslant t \leqslant \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

CAS **29–31** Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

29. $x = t, \; y = e^{-t}, \; z = 2t - t^2; \quad (0, 1, 0)$

30. $x = 2 \cos t, \; y = 2 \sin t, \; z = 4 \cos 2t; \quad (\sqrt{3}, 1, 2)$

31. $x = t \cos t, \; y = t, \; z = t \sin t; \quad (-\pi, \pi, 0)$

32. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.
(b) Illustrate by graphing the curve and both tangent lines.

33. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

Graphing calculator or computer required CAS Computer algebra system required **1.** Homework Hints available at stewartcalculus.com

34. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.

35–40 Evaluate the integral.

35. $\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt$

36. $\int_0^1 \left(\frac{4}{1+t^2}\,\mathbf{j} + \frac{2t}{1+t^2}\,\mathbf{k} \right) dt$

37. $\int_0^{\pi/2} (3 \sin^2 t \cos t\,\mathbf{i} + 3 \sin t \cos^2 t\,\mathbf{j} + 2 \sin t \cos t\,\mathbf{k})\,dt$

38. $\int_1^2 (t^2\,\mathbf{i} + t\sqrt{t-1}\,\mathbf{j} + t \sin \pi t\,\mathbf{k})\,dt$

39. $\int (e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k})\,dt$

40. $\int (\cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j} + t\,\mathbf{k})\,dt$

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41. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

42. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + te^t\,\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

43. Prove Formula 1 of Theorem 3.

44. Prove Formula 3 of Theorem 3.

45. Prove Formula 5 of Theorem 3.

46. Prove Formula 6 of Theorem 3.

47. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

48. If $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 47, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$$

49. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

50. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 49, find $\mathbf{r}'(2)$.

51. Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

52. Find an expression for $\frac{d}{dt}[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

53. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

[*Hint:* $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$]

54. If a curve has the property that the position vector $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$, show that the curve lies on a sphere with center the origin.

55. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

56. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t. [*Hint:* Refer to Figure 1 and consider the cases $h > 0$ and $h < 0$ separately.]

13.3 Arc Length and Curvature

In Section 10.2 we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$\boxed{1} \qquad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

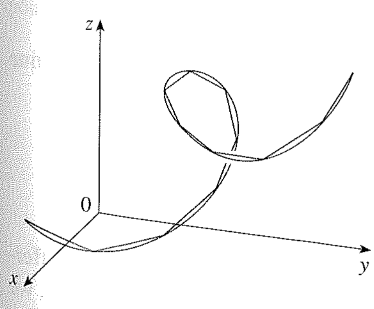

FIGURE 1
The length of a space curve is the limit of lengths of inscribed polygons.

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leqslant t \leqslant b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\boxed{2} \qquad \begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \end{aligned}$$