METU - NCC

<table>
<tr><td colspan="2">Calculus for Functions of Several Variables
Midterm</td><td></td></tr>
<tr><td>Code : Math 120
Acad.Year: 2011-2012
Semester : Summer
Date : 25.7.2012
Time : 17:40
Duration : 120 minutes</td><td colspan="2">Last Name:
Name : Student No.:
Department: Section:
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7 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS</td></tr>
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1 (13)	2 (15)	3 (8)	4 (15)	5 (16)	6 (18)	7 (15)		

1. (5 +5 +3=13 pts) (a) Write an equation of the plane $\mathcal{P}$ that contains the line of intersection of the planes $x = 0$ and $z = 0$, and passing through the point $(1, 1, 1)$.

$x=0$ is the yz-plane $z=0$ is the xy-plane. Their intersection is the y-axis. with equation $r(t) = \langle 0,t,0\rangle = \langle 0,0,0\rangle + t\langle 0,1,0\rangle$
$\vec{u} = \langle 1,1,1\rangle$ will be on the plane.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1i - 1k \qquad 1\cdot(x-0) + 0\cdot(y-0) + (-1)(z-0) = 0$$

$$\boxed{z = x}$$

(b) Write an equation of ~~the~~ a line $\mathcal{L}$ which is parallel to the plane $\mathcal{P}$ and passing through $(2, 1, 1)$.

Direction vector of $\mathcal{L}$ must be perpendicular to $\vec{n} = \langle 1,0,-1\rangle$
We'll have infinitely many such lines. Let's choose $\vec{u} = \langle 1,1,1\rangle$
$\mathcal{L}$ will have equation $\ell(t) = \langle 2,1,1\rangle + t\langle 1,1,1\rangle$

(c) Find the distance between the plane $\mathcal{P}$ and the line $\mathcal{L}$.

Since $\mathcal{L}$ is parallel to $\mathcal{P}$, distance of any point on the line $\mathcal{L}$ to $\mathcal{P}$ will work. Take $Q = (2,1,1)$, and compute the distance

$$\text{Distance} = \frac{|2-1|}{\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}}$$

2. (5 +5 +5=15 pts) Determine whether or not the following limits exist. If they exist, then find the limit. Explain your answer.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{xy + x - y}$

Along $x=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{xy+x-y} = \lim_{y\to 0} \frac{0}{-y} = 0$ $\not\parallel$

Along $y=x$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{xy+x-y} = \lim_{x\to 0} \frac{x^2}{x^2+x-x} = \lim_{x\to 0} 1 = 1$

<u>No Limit.</u>

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 \sin^4 y}{x^4 + 4y^4} = 0$ by Squeeze Thm (below)

$$0 \le \left(\frac{x^4}{x^4+4y^4}\right) \cdot \sin^4 y \le \sin^4 y$$

as $(x,y)\to(0,0)$ $\qquad$ by Squeeze Thm $\qquad$ as $(x,y)\to(0,0)$

$$0$$

c. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

Along $x=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{0}{y^6} = 0$ $\not\parallel$

Along $x=y^3$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{y^6}{y^6+y^6} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$

<u>No Limit</u>

3. (8 pts) Find the directional derivative of the function $f(x, y, z) = \frac{x}{y+z}$ at $(1, 1, 0)$ in the direction for which the function $g(x, y, z) = \ln(x^2 + y^2 + z^2)$ increases most rapidly at the same point.

$g(x,y,z)$ increases most rapidly in $\nabla g(1,1,0)$ - direction.

$$\nabla g\Big|_{(1,1,0)} = \left\langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right\rangle\Big|_{(1,1,0)} = \langle 1,1,0 \rangle$$

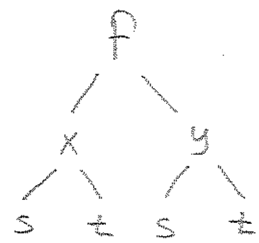
$$D_{\langle 1,1,0\rangle} f = \nabla f(1,1,0) \cdot \frac{\langle 1,1,0\rangle}{|\langle 1,1,0 \rangle|} = \langle 1,-1,-1\rangle \cdot \frac{\langle 1,1,0\rangle}{\sqrt{2}} = 0$$

$$\nabla f\Big|_{(1,1,0)} = \left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2} \right\rangle\Big|_{(1,1,0)} = \langle 1,-1,-1 \rangle$$

4. (8+7 =15 pts) Suppose $f = f(x, y)$ is a function with continuous second order partial derivatives with $x = e^s t$, $y = se^t$. Given the following table of values:

$f(0,1) = -2$	$f_x(0,1) = 2$	$f_y(0,1) = 4$	$f(1,0) = -1$	$f_x(1,0) = 3$	$f_y(1,0) = 2$
$f_{xx}(0,1) = 3$	$f_{xy}(0,1) = 1$	$f_{yy}(0,1) = -1$	$f_{xx}(1.0) = -1$	$f_{xy}(1,0) = 1$	$f_{yy}(1,0) = 3$

a. Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $(s, t) = (1, 0)$.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = f_x \cdot e^s t + f_y \cdot e^t$$

$$\frac{\partial f}{\partial s}(1,0) = f_x(0,1)\cdot e^1 \cdot 0 + f_y(0,1)\cdot e^0 = 4$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = f_x \cdot e^s + f_y\, s\, e^t$$

$$\frac{\partial f}{\partial t}(1,0) = f_x(0,1)\cdot e^1 + f_y(0,1)\cdot 1 \cdot e^0 = 2e+4$$

b. $\frac{\partial^2 f}{\partial s \partial t}$ at $(s, t) = (1, 0)$.

$$\frac{\partial^2 f}{\partial s\,\partial t} = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) = \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial s}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\left[\frac{\partial^2 x}{\partial s\,\partial t}\right]$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial y}{\partial s}\right]\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y}\cdot\left[\frac{\partial^2 y}{\partial s\,\partial t}\right]$$

At $(s,t) = (1,0)$

$$\frac{\partial^2 f}{\partial s\,\partial t}(1,0) = \left(f_{xx}(0,1)\cdot e^1 \cdot 0 + f_{xy}(0,1)\cdot e^0\right)\cdot e^1 + f_x(0,1)\cdot e^1$$

$$+ \left(f_{yx}(0,1)\cdot e^1 \cdot 0 + f_{yy}(0,1)\cdot e^0\right)\cdot 1 \cdot e^0 + f_y(0,1)\cdot e^0 = e + 2e - 1 + 4$$

$$= 3e + 3$$

5. (8+5+3=16 pts)Given $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

a. Find and classify the critical points of $f(x,y)$.

$f_x = 6xy - 6x = 6x(y-1) = 0 \Rightarrow x=0$ OR $y=1$

$f_y = 3x^2 + 3y^2 - 6y = 0 \Rightarrow$

$x=0: 3y^2-6y=0$, $3y(y-2)=0$, $y=0$ or $y=2$

$y=1: 3x^2-3=0$, $3(x^2-1)=0$, $x=1$ or $x=-1$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{bmatrix}$$

Critical Points are $(0,0), (0,2), (1,1), (-1,1)$

$(0,0)$: $\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ $D_2 = (-6)\cdot(-6) = 36 > 0$, $D_1 = -6 < 0$

Local Maximum

$(0,2)$: $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ $D_2 = 6\cdot 6 = 36 > 0$, $D_1 = 6 > 0$

Local minimum

$(1,1)$: $\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$ $D_2 = 0 - 6\cdot 6 = -36 < 0$

Saddle

$(-1,1)$: $\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$ $D_2 = 0 - (-6)(-6) = -36 < 0$

Saddle

b. Find an equation of the tangent plane to the surface $z = f(x,y)$ at the point $(x,y) = (1,0)$.

$f_x(1,0) = -6$ $f_y(1,0) = 3$ $f(1,0) = -1$

$$-6(x-1) + 3(y-0) - (z+1) = 0$$

$$z = -1 - 6(x-1) + 3y$$

c. Using the point in part (b) approximate $f(1.1, -0.2)$.

Linearization at $(1,0)$ is $L(x,y) = -1 - 6(x-1) + 3y$

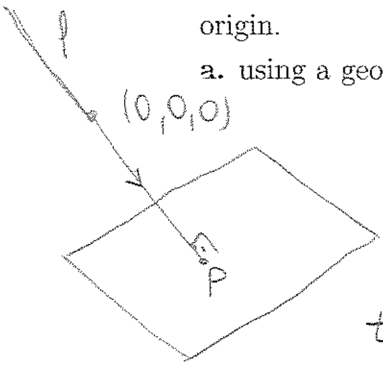
$$f(1.1,-0.2) \approx L(1.1,-0.2) = -1 - 6(0.1) + 3\cdot(-0.2)$$

$$= -1 - 0.6 - 0.6$$

$$= \underline{\underline{-2.2}}$$

6. (4+6+8=18 pts) Find the point on the plane $x + 2y + 2z = 3$ which is closest to the origin.

a. using a geometric argument. (No calculus)



Closest point P and origin must be on a line in the direction of normal vector of the plane.

$\ell(t) = \langle 0,0,0\rangle + t\langle 1,2,2\rangle = \langle t, 2t, 2t\rangle$

$t + 2(2t) + 2(2t) = 3 \Rightarrow t = \frac{1}{3}$

so $P = \ell(\frac{1}{3}) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

b. by reducing the problem to an unconstrained problem of two variables.

We need to minimize the distance $f(x,y,z) = (x-0)^2 + (y-0)^2 + (z-0)^2$

$z = \frac{1}{2}(3-x-2y)$ implies $f(x,y) = x^2 + y^2 + \frac{1}{4}(3-x-2y)^2$

$$\left.\begin{aligned} f_x &= 2x + \tfrac{1}{2}(3-x-2y)(-1) = \tfrac{5}{2}x + y - \tfrac{3}{2} = 0 \\ f_y &= 2y + \tfrac{1}{2}(3-x-2y)(-2) = x + 4y - 3 = 0 \end{aligned}\right\} \quad x = \frac{1}{3},\ y = \frac{2}{3}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 1 \\ 1 & 4 \end{bmatrix} \quad \left.\begin{aligned} D_2 &= 10 - 1 = 9 > 0 \\ D_1 &= \tfrac{5}{2} > 0 \end{aligned}\right\}$$

Local minimum, but only critical point, so global minimum

$\left(x, y, \frac{1}{2}(3-x-2y)\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

c. using the method of Lagrange multipliers.

$f(x,y,z) = x^2 + y^2 + z^2$ with constraint $x + 2y + 2z = 3$

$\nabla f = \lambda \nabla g \Rightarrow$ (i) $2x = \lambda \Rightarrow x = \frac{\lambda}{2}$

(ii) $2y = 2\lambda$ $\quad y = \lambda$

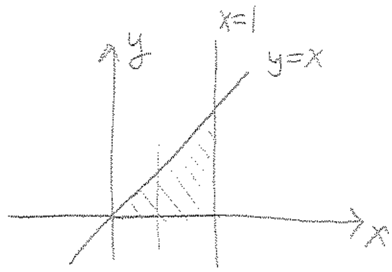
(iii) $2z = 2\lambda$ $\quad z = \lambda$

(iv) $x + 2y + 2z = 3$ $\quad \frac{\lambda}{2} + 2\lambda + 2\lambda = 3 \Rightarrow \lambda = \frac{2}{3}$

Hence, $(x,y,z) = \left(\frac{2/3}{2}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

7. (5+5+5=15 pts) This problem has three unrelated parts.

a. Evaluate $\int_0^1 \int_y^1 \frac{xy}{1+x^4}\,dx\,dy$



Switch order: $\displaystyle\int_0^1\int_0^x \frac{xy}{1+x^4}\,dy\,dx$

$$= \int_0^1 \left(\frac{1}{2}\, \frac{x \cdot y^2}{1+x^4} \Big|_0^x \right) dx = \frac{1}{2}\int_0^1 \frac{x^3}{1+x^4}\,dx = \frac{1}{8}\int_1^2 \frac{1}{u}\,du = \frac{1}{8}\ln|u| \Big|_1^2$$

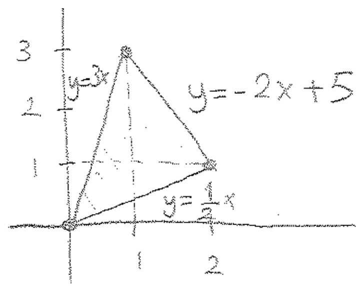
$u = 1+x^4$

$du = 4x^3 dx$

$$= \frac{1}{8}\ln 2 - \frac{1}{8}\underbrace{\ln 1}_{0}$$

$$= \frac{1}{8}\ln 2$$

b. Write the iterated double integrals which compute the area of the triangle with vertices (0,0), (1,3), and (2,1).



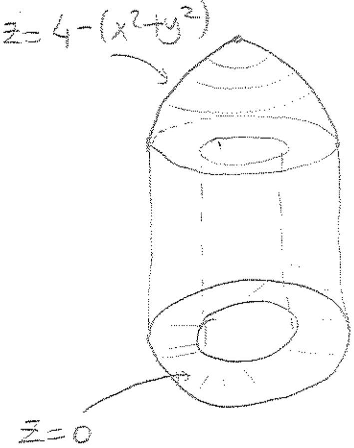
(i) $dydx$ order

$$\int_0^1\int_{\frac{1}{2}x}^{3x} 1\,dy\,dx + \int_1^2\int_{\frac{1}{2}x}^{-2x+5} 1\,dy\,dx$$

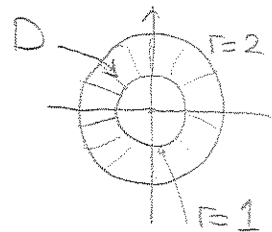
(ii) $dxdy$ order

$$\int_0^1\int_{\frac{y}{3}}^{2y} 1 \cdot dx\,dy + \int_1^3\int_{\frac{y}{3}}^{\frac{5-y}{2}} 1 \cdot dx\,dy$$

c. Find the volume of the solid that lies between the cylinders $x^2+y^2=1$, $x^2+y^2=4$ and bounded by the surfaces $z=0$, $z=4-(x^2+y^2)$



$$\iint_D \left(4-(x^2+y^2)-0\right) dy\,dx$$



$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2} (4-r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_1^2 (4r - r^3)\,dr$$

polar coordinates

$$= 2\pi \cdot \left(2r^2 - \frac{r^4}{4}\right)\Big|_1^2 = 2\pi \cdot \frac{9}{4}$$