M E T U – N C C
Mathematics Group

Calculus with Analytic Geometry First Midterm Exam		
Code : MATH 119 Acad.Year : 2011 Semester : Fall Coord. : S.D/I.U/H.T.	Last Name : Name : Dept. : Signature :	Stud. No : Sec. No :
Date : 30.10.2011 Time : 9.40 Duration : 120 minutes	8 Questions on 6 Pages Total 100 Points	

P1	P2	P3	P4	P5	P6		

Q.1 (5 × 3 = 15 pts) Evaluate the following derivatives:

(a) If $f(x) = \dfrac{\cos x}{1 + \sin x}$, find $f'(0)$.

$$f'(x) = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$f'(0) = -1$$

(b) If $g(\theta) = \cos(5\theta^2 + \theta \sin\theta)$, find $g'(\theta)$.

$$g'(\theta) = -\sin(5\theta^2 + \theta.\sin\theta).(10\theta + \sin\theta + \theta.\cos\theta)$$

(c) If $z = \left(u + \dfrac{1}{u-1}\right)^{-5/3}$, find $\dfrac{dz}{du}$.

$$\frac{dz}{du} = \frac{-5}{3}\left(u + \frac{1}{u-1}\right)^{-8/3}\left(1 - \frac{1}{(u-1)^2}\right)$$

Q.2 **(3 × 5 = 15 pts)** Evaluate the following limits:

(a) $\lim_{x \to \pi/2} \frac{x - x\sin x}{1 - \sin^2 x}$ $= \lim_{x \to \pi/2} \frac{x(1-\sin x)}{(1-\sin x)(1+\sin x)}$

$= \lim_{x \to \pi/2} \frac{x}{1+\sin x} = \pi/4$

(b) $\lim_{x \to \pi/2} \frac{x \sin x}{1 + \cos x}$ $= \frac{\pi/2 \cdot \sin \pi/2}{1 + \cos \pi/2} = \pi/2$

(c) $\lim_{x \to 1} \frac{x^3 - 1}{|x^2 - 1|}$ $\lim_{x \to 1^+} \frac{x^3-1}{|x^2-1|} = \lim_{x \to 1^+} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{3}{2}$

$\lim_{x \to 1^-} \frac{x^3-1}{|x^2-1|} = \lim_{x \to 1^-} \frac{(x-1)(x^2+x+1)}{-(x-1)(x+1)} = -\frac{3}{2}$

So $\lim_{x \to 1} \frac{x^3-1}{|x^2-1|}$ does not exist

(d) $\lim_{\theta \to 0} \theta \cot(2\theta)$ $= \lim_{\theta \to 0} \frac{\theta \cdot \cos 2\theta}{\sin 2\theta} = \lim_{\theta \to 0} \cos 2\theta \cdot \frac{1}{2} \cdot \frac{2\theta}{\sin 2\theta}$

$= \frac{1}{2}$

(e) $\lim_{x \to 0} x^4 \sin\left(\frac{\pi}{x^4}\right)$ $-1 \le \sin\left(\frac{\pi}{x^4}\right) \le 1 \Rightarrow -x^4 \le x^4 \sin\left(\frac{\pi}{x^4}\right) \le x^4$

Since $\lim_{x \to 0} -x^4 = \lim_{x \to 0} x^4 = 0$, by squeeze theorem

$\lim_{x \to 0} x^4 \sin(\pi/x^4) = 0$

Q.3 (10 pts) Let $f(x) = \begin{cases} x^2 + ax + 3 & , \quad x \leq 1 \\ bx^2 + 2x - 1 & , \quad x > 1 \end{cases}$ be a function in which a and b are some constants. Find the values of a and b so that $f(x)$ is differentiable everywhere.

For $f(x)$ to be cont. we need

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) = a + 4$$

So $b + 1 = a + 4 \Rightarrow b = a + 3 \quad (*)$

For $f(x)$ to be diff. we need the derivative of $f(x)$ from the left & right to be equal: (at $x = 1$)

$$2 + a = 2b + 2$$

with $(*)$:

$$2 + a = 2(a+3) + 2$$

$$2 + a = 2a + 8$$

$$a = -6 \qquad b = -3$$

Q.4 (10 pts) Prove that

$$\lim_{x \to 1} \frac{6x}{2x+1} = 2$$

using ϵ, δ definition of limit.

We want for all $\varepsilon > 0$, there is $\delta > 0$ s.t.

$$0 < |x - 1| < \delta \Rightarrow \left| \frac{6x}{2x+1} - 2 \right| < \varepsilon$$

$$\left| \frac{6x}{2x+1} - 2 \right| = \left| \frac{6x - 4x - 2}{2x+1} \right| = \frac{2|(x-1)|}{|2x+1|}$$

As $x \to 1$, we may assume $x \in (0,2)$. i.e. $\delta \leq 1$

, then $|2x+1| \geq 1 \Rightarrow \dfrac{1}{|2x+1|} \leq 1 \quad (*)$

Let $\varepsilon > 0$ & $\delta = \min\{\varepsilon/2, 1\}$ Then

$$|x-1| < \delta \Rightarrow |x - 1| < \frac{\varepsilon}{2} \Rightarrow 2|x-1| < \varepsilon \overset{\text{By } (*)}{\Rightarrow} \frac{2|x-1|}{|2x+1|} < \varepsilon$$

$$\Rightarrow \left| \frac{6x}{2x+1} - 2 \right| < \varepsilon$$

Q.5 **(15 pts)** Find maximum and minimum values of $f(x) = (x^2+1)(x^2-1)^{2/3}$ on $x \in [0, \sqrt{2}]$.

$$f'(x) = 2x\,(x^2-1)^{2/3} + \frac{2}{3}\frac{(x^2+1).2x}{(x^2-1)^{1/3}}$$

$$f'(x) = \frac{6x\,(x^2-1) + 2(x^2+1).2x}{3\,(x^2-1)^{1/3}}$$

$$f'(x) = \frac{6x^3 - 6x + 4x^3 + 4x}{3\,(x^2-1)^{1/3}}$$

$$f'(x) = \frac{10x^3 - 2x}{3\,(x^2-1)^{1/3}} \Rightarrow$$ Crit Points : $x = 0$, $x = \mp \frac{1}{\sqrt{5}}$, $x = \mp 1$

Since $x \in [0, \sqrt{2}]$, we look at

$f(0) = 1$

$f\left(\frac{1}{\sqrt{5}}\right) = \frac{6}{5} \cdot \left(\frac{-4}{5}\right)^{2/3}$

$f(1) = 0 \longleftarrow$ Min

$f(\sqrt{2}) = 3 \longleftarrow$ Max

Q.6 **(5 + 5 = 10 pts)**

(a) Approximate the number $\cos\left(\frac{\pi}{4} + 0.1\right)$ using a linear approximation.

At $x = \pi/4$, $f(x) = \cos x$ has tangent

$$L(x) = \cos(\pi/4) - \sin(\pi/4)\,(x - \pi/4) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,(x - \pi/4)$$

$$\cos(\pi/4 + 0.1) \cong L(\pi/4 + 0.1)\ \frac{1}{\sqrt{2}} - \frac{1}{10\sqrt{2}}$$

(b) Using differentials, estimate the amount of paint needed to apply a coat of paint $0.05cm$ thick to a cubic box side lengths $10cm$.

$V = x^3$, V = volume , x = side length

$\Delta V \simeq dV = 3x^2 . dx$. When $x = 10$ & $dx = 0.05$:

$$\Delta V \simeq 3.100.(0.05) = \frac{300.5}{100} = 15\ cm^3$$

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Q.7 (10 + 5 = 15 pts) Consider the plane curve $\mathcal{C}$ defined implicitly by the equation

$$x^3 + \cos y - y \sin x = 0.$$

(a) Verify that the point $P(x, y) = P(0, \frac{\pi}{2})$ lies on $\mathcal{C}$. Then find the equations for the lines that are tangent and normal to $\mathcal{C}$ at P.

$$0^3 + \cos(\pi/2) - \pi/2 \cdot \sin(0) = 0 \checkmark$$

$$3x^2 - \sin y \cdot \frac{dy}{dx} - \frac{dy}{dx} \sin x - y \cdot \cos x = 0$$

$$3x^2 - y \cdot \cos x = \sin y \frac{dy}{dx} + \sin x \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2 - y\cos x}{\sin y + \sin x}$$

$$\left.\frac{dy}{dx}\right|_{(0,\pi/2)} = \frac{-\pi/2}{1} = -\pi/2$$

Tangent: $\quad y - \pi/2 = -\pi/2\,(x - 0)$

Normal: $\quad y - \pi/2 = \frac{2}{\pi}\,(x - 0)$

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d^2y}{dx^2} = \frac{\left(6x - \frac{dy}{dx} \cdot \cos x + y \sin x\right)(\sin y + \sin x) - (3x^2 - y\cos x)\left(\cos y \cdot \frac{dy}{dx} + \cos x\right.}{(\sin y + \sin x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left[6x - \frac{3x^2 - y\cos x}{\sin y + \sin x} \cdot \cos x + y\sin x\right](\sin y + \sin x) - (3x^2 - y\cos x)\left(\cos y \, \frac{3x^2 - y\cos x}{\sin y + \sin x} + c\right.}{(\sin y + \sin x)^2}$$

Q.8 **(10 pts)** The shaded area A in the Figure, given by the formula

$$A = \frac{2}{3}(x+1)(y+1) - x$$

denotes an increasing function of time t, as the point $P(x, y)$ moves ahead along the curve C. Find the rate of change of A if x and y coordinates of P are both increasing at a rate of $1cm/min$, when $A = 10cm^2$ and $y = 2cm$.

$$\frac{dA}{dt} = \frac{2}{3}\frac{dx}{dt}(y+1) + \frac{2}{3}(x+1)\cdot\frac{dy}{dt} - \frac{dx}{dt}$$

$$\left.\frac{dA}{dt}\right|_{\substack{A=10\\ y=2}} = \frac{2}{3}\cdot 1\cdot 3 + \frac{2}{3}\cdot 9\cdot 1 - 1 = 7$$

$$\left[\begin{array}{ll} \text{when } A=10 & 10 = \frac{2}{3}(x+1)\cdot 3 - x \\ \quad\quad\;\; y=2 & 10 = 2x+2-x \\ & x = 8 \end{array}\right]$$