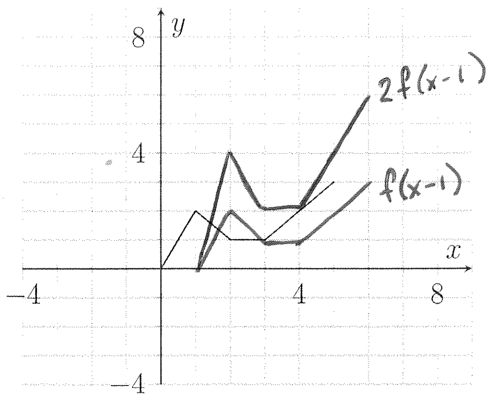
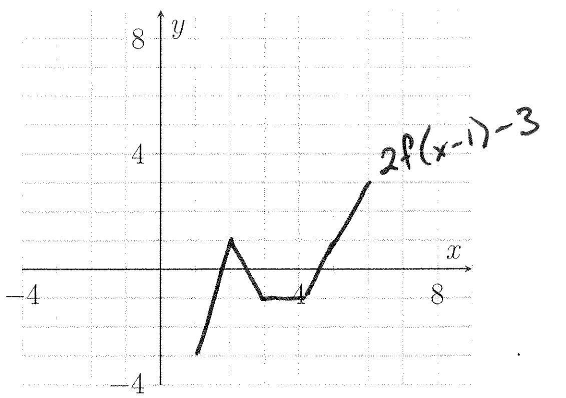
	Calculus and Analytical Geometry I. Midterm	
Code : *Math 119* Acad.Year : *2010-2011* Semester : *Fall* Date : *1.11.2010* Time : *17:40* Duration : *120 minutes*	Last Name: Name : Solutions Student No: Department: Section: Signature:	
	9 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	8	9		

1.(5 pts) The graph of the function $f(x)$ is given below. Using this graph, plot the graph of $2f(x-1)-3$ in the empty coordinate plane at the bottom.





2. (5+5+5+5 pts) Evaluate the following limits, if they exist. Show your work. Do not use L'Hospital's rule.

(a) $\lim_{x\to 2} \frac{|x^2-4|}{x-2}$

$$\frac{|x^2-4|}{x-2} = \begin{cases} \frac{x^2-4}{x-2} & x \geq 2 \text{ or } x \leq -2 \\ \frac{4-x^2}{x-2} & -2 \leq x \leq 2 \end{cases}$$

$$\lim_{x\to 2^+} \frac{|x^2-4|}{x-2} = \lim_{x\to 2^+} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} x+2 = 4$$

$$\lim_{x\to 2^-} \frac{|x^2-4|}{x-2} = \lim_{x\to 2^-} \frac{4-x^2}{x-2} = \lim_{x\to 2^-} -(x-2) = -4$$

Since these are not equal, $\lim_{x\to 2}$ does not exist!

(b) $\lim_{x\to 0+} x^2 \sin(1/x) + \frac{1}{\sqrt{x}} \sin(x)$

$$-x^2 \leq x^2 \sin(1/x) \leq x^2$$

Since $\lim_{x\to 0^+} -x^2 = \lim_{x\to 0^+} x^2 = 0$ by the squeeze theorem, $\lim_{x\to 0^+} x^2 \sin(1/x) = 0$.

Also $\lim_{x\to 0^+} \frac{\sin(x)}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x} \frac{\sin x}{x} = 0 \cdot 1 = 0$

So $\lim_{x\to 0^+} x^2 \sin(1/x) + \frac{1}{\sqrt{x}} \sin(x) = 0$.

(c) $\lim_{x\to 2} \frac{\sqrt{1+x^3}-3}{x^2-2x-3}$

Note that $\lim_{x\to 2} x^2-2x-3 = -3 \neq 0$

Thus, $\lim_{x\to 2} \frac{\sqrt{1+x^3}-3}{x^2-2x-3} = \frac{0}{-3} = 0$.

(d) $\lim_{x\to 0+} \frac{x}{\sqrt{1-\cos(x)}}$

$$\lim_{x\to 0^+} \frac{x}{\sqrt{1-\cos(x)}} = \lim_{x\to 0^+} \frac{x}{\sqrt{1-\cos x}} \cdot \sqrt{\frac{(1+\cos x)}{(1+\cos x)}}$$

$$= \lim_{x\to 0^+} x\sqrt{\frac{1+\cos x}{1-\cos^2 x}}$$

$$= \lim_{x\to 0^+} \frac{x}{\sqrt{\sin^2 x}} \sqrt{1+\cos x}$$

$$= \lim_{x\to 0^+} \frac{x}{\sin x} \sqrt{1+\cos x} = 1 \cdot \sqrt{2} = \sqrt{2}.$$

3. (10 pts) Can a function satisfying $f(x+h) - f(x) = \sqrt[3]{h}$ be differentiable?
(Explain your answer.)

$$f(x+h) - f(x) = h^{1/3}.$$

So $$\frac{f(x+h) - f(x)}{h} = h^{-2/3} \quad \text{for } h \neq 0.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} h^{-2/3}$$ which does not exist!

Hence $f(x)$ is not differentiable at any x.

4. (5+5+5 pts) Find the indicated derivatives. (You do not need to simplify your answer.)

(a) $f(x) = \dfrac{(3x+1)^4}{x\sec(x)}$. Find $f'(x)$.

$$f'(x) = \frac{\left((3x+1)^4\right)' x\sec x - (3x+1)^4 (x\sec x)'}{x^2\sec^2 x}$$

$$= \frac{4(3x+1)^3\cdot 3\, x\sec x - (3x+1)^4(\sec x + x\sec x\tan x)}{x^2\sec^2 x}$$

(b) $f(x) = \sec^2\left(\sin(3x+1)\right)$. Find $f'(x)$.

$$f'(x) = 2\sec(\sin(3x+1))\cdot\sec(\sin(3x+1))\tan(\sin(3x+1))\cdot\cos(3x+1)\cdot 3$$

$$= 6\sec^2(\sin(3x+1))\cdot\tan(\sin(3x+1))\cdot\cos(3x+1).$$

(c) Compute $\dfrac{d^2}{dx^2}\Big(f(g(x))\Big)$ at $x=3$, given that

$g(3)=5$	$f(3)=0$	$f(5)=1$
$g'(3)=1$	$f'(3)=2$	$f'(5)=3$
$g''(3)=-1$	$f''(3)=-2$	$f''(5)=4$

$$(f\circ g)' = (f'\circ g)\cdot g'$$

$$(f\circ g)'' = \left((f'\circ g)\cdot g'\right)'$$

$$= (f'\circ g)'\cdot g' + (f'\circ g)\cdot g''$$

$$= (f''\circ g)(g')^2 + (f'\circ g)\cdot g''$$

$$(f\circ g)''(3) = f''(g(3))\cdot(g'(3))^2 + f'(g(3))\cdot g''(3)$$

$$= f''(5)\cdot 1^2 + f'(5)\cdot(-1)$$

$$= 4\cdot 1 + 3\cdot(-1) = \underline{\underline{1.}}$$

5. (10 pts) Find the equation of the tangent line to the curve $x^2 + xy + y^2 = 1$ at the point $(1, -1)$.

$$x^2 + xy + y^2 = 1$$

$\frac{d}{dx}$ Implicit differentiation with respect to x

$$2x + (y + xy') + 2yy' = 0$$

At the point $(1,-1)$ this is

$$2 + (-1 + y') + 2(-1)y' = 0$$

$$\Longrightarrow y' = 1$$

This is the slope of the tangent line at $(1,-1)$.

Equation of tangent line:

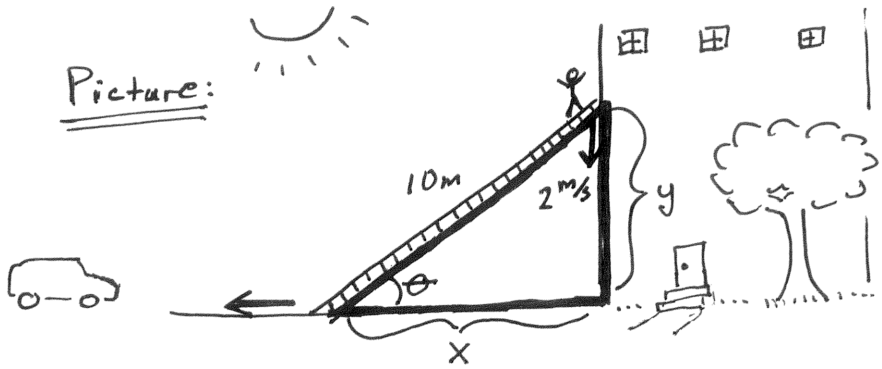
$$y = y'(x_0)\,(x - x_0) + y(x_0)$$

$$y = 1\,(x-1) + (-1)$$

$$\boxed{y = x - 2}$$

6. (10 pts) A 10 meter ladder is leaning on a wall and starts to slide down with its bottom end on the ground and top end on the wall. The top of the ladder is sliding down towards the ground at a speed of $2m/s$. Let θ be the angle between the ground and the ladder. At what rate is the angle θ changing when the top of the ladder is 6 meters away from the ground?

Picture:



$$\frac{d}{dt}y = -2 \text{ m/s} \qquad \frac{d}{dt}\theta = ?? \text{ rad/s} \quad \text{when } y = 6\text{ m}$$

$$\sin\theta = \frac{y}{10}$$

$$y = 10\sin\theta$$

$$\Big\downarrow \frac{d}{dt}$$

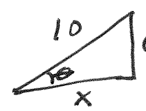
$$\underbrace{\frac{d}{dt}y}_{-2\text{ m/s}} = 10\cos\theta\left(\frac{d}{dt}\theta\right)$$

$$\frac{d}{dt}\theta = \frac{-1}{5\cos\theta}$$

$$= \frac{-1}{5\cdot 4/5}$$

$$= \boxed{-\tfrac{1}{4} \text{ rad/s}}$$

When $y = 6$ m,



$$x^2 + 36 = 100$$

$$x = 8 \text{ m}$$

$$\cos\theta = x/10 = 8/10 = 4/5$$

7. (10 pts) Find the absolute minimum and absolute maximum of the function $f(x) = x^2 - 6|x-1|$ on the interval $[-4, 5]$.

$f(x)$ is continuous on $[-4, 5]$, so by the extreme value theorem, it has an absolute minimum and an absolute maximum.

Candidates for absolute min/max:

1.) Endpoints: -4, 5

2.) Points where $f'(x)$ does not exist: 1,

3.) Points where $f'(x) = 0$:

$$f(x) = \begin{cases} x^2 - 6x + 6 & x \geq 1 \\ x^2 + 6x - 6 & x < 1 \end{cases}$$

$\underline{\underline{x > 1}}$: $f(x) = x^2 - 6x + 6$

$f'(x) = 2x - 6$

$f'(x) = 0 \iff x = -3$

$\underline{\underline{x < 1}}$: $f(x) = x^2 + 6x - 6$

$f'(x) = 2x + 6$

$f'(x) = 0 \iff x = 3$

$f(-4) = 16 - 6|-4-1| = -14$

$f(-3) = 9 - 6|-3-1| = -15$ absolute minimum.

$f(1) = 1 - 6|1-1| = 1$

$f(3) = 9 - 6|3-1| = -3$ absolute maximum.

$f(5) = 25 - 6|5-1| = 1$

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8. (10 pts) Use *linear approximation* to estimate the value of $\sqrt{65} + \sqrt[3]{65}$.

Let $f(x) = \sqrt{x} + \sqrt[3]{x}$.

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3}$$

$$f(64) = \sqrt{64} + \sqrt[3]{64} = 8 + 4 = 12.$$

$$f'(64) = \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{3}\cdot\frac{1}{16} = \frac{4}{3\cdot 16} = 1/12$$

Near 64, by linear approximation,

$$f(x) \approx f'(64)(x-64) + f(64)$$

$$= \frac{1}{12}(x-64) + 12$$

$$f(65) \approx \boxed{\frac{1}{12} + 12}$$

9. (10 pts) Show that $f(x) = x^4 - 4x^2 - 3x + 10$ has exactly 3 critical points.

Since $f(x)$ is differentiable everywhere, its only critical points will be the places where $f'(x) = 0$.

$$f'(x) = 4x^3 - 8x - 3$$

$f'(x)$ is a cubic polynomial, so it has at most three real roots. It is enough to show that it has at least three real roots. Note that f' is cont. so IVT is ok.

$$f'(-10) = -4000 - 80 - 3 < 0$$

$$f'(-1) = -4 + 8 - 3 > 0$$

$$f'(0) = -3 < 0$$

$$f'(10) = 4000 - 80 - 3 > 0$$