M E T U
Northern Cyprus Campus

	Introduction to Differential Equations Midterm II	

Code : *Math 219* Acad.Year: *2010-2011* Semester : *Summer* Date : *02.8.2011* Time : *18:30* Duration : *120 minutes*	Last Name: Name : Student No: Department: Section: Signature: 6 QUESTIONS ON 6 PAGES TOTAL 105 POINTS

1	2	3	4	5	6		

Show your work! No calculators! Please draw a box **around your answers!**
Please do not write on your desk!

1. (15) Find the solution to the initial value problem

$$y''' - 2y'' + y' - 2y = 0 \qquad y(0) = 1, y'(0) = -1, y''(0) = 0$$

Characteristic polynomial of the D. Eqn is

$r^3 - 2r^2 + r - 2 = r^2(r-2) + r - 2 = (r-2)(r^2+1)$

Its roots are $r_1 = 2$, $r_2 = +i$, $r_3 = -i$

Hence, the general solution is $y(t) = c_1 . e^{2t} + c_2 . \cos t + c_3 . \sin t$

$y'(t) = 2c_1 . e^{2t} - c_2 . \sin t + c_3 . \cos t$

$y''(t) = 4c_1 . e^{2t} - c_2 . \cos t - c_3 . \sin t$

$$\left.\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ -1 &= y'(0) = 2c_1 + c_3 \\ 0 &= y''(0) = 4c_1 - c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 + c_2 &= 1 \\ + \quad 4c_1 - c_2 &= 0 \\ \hline 5c_1 &= 1 \end{aligned}$$

$c_1 = \frac{1}{5}$, $c_2 = \frac{4}{5}$

$c_3 = -\frac{7}{5}$

$$\boxed{\therefore\ y(t) = \frac{1}{5} e^{2t} + \frac{4}{5} . \cos t - \frac{7}{5} . \sin t}$$

2.(9+8+9) This problem has two unrelated parts.

(a) Find the inverse Laplace transform of the function $F(s) = \dfrac{e^{-11s}s}{s^2 - 2s + 5}$

$$F(s) = e^{-11s} \cdot \frac{s}{s^2 - 2s + 5} = e^{-11s} \cdot \frac{s}{(s-1)^2 + 2^2}$$

$$= e^{-11s} \cdot \frac{(s-1)+1}{(s-1)^2 + 2^2} = \frac{e^{-11s} \cdot (s-1)}{(s-1)^2 + 2^2} + \frac{1}{2} \cdot e^{-11s} \frac{2}{(s-1)^2 + 2^2}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = U_{11}(t) \cdot e^{(t-11)} \cdot \cos(2(t-11)) + \frac{1}{2} \cdot U_{11}(t) \cdot e^{(t-11)} \cdot \sin(2(t-11))}$$

(b) Compute $e^{2t} * e^{3t}$

1st Method: $e^{2t} * e^{3t} = \int_0^t e^{2(t-\tau)} e^{3\tau} d\tau = e^{2t} \int_0^t e^{-2\tau} \cdot e^{3\tau} d\tau$

$$= e^{2t} \cdot \int_0^t e^{\tau} d\tau = e^{2t} \left(e^{\tau} \Big|_0^t \right) = e^{2t}(e^t - 1) = e^{3t} - e^{2t}$$

2nd Method: $\mathcal{L}\{e^{2t} * e^{3t}\} = \frac{1}{s-2} \cdot \frac{1}{s-3} = \frac{-1}{s-2} + \frac{1}{s-3}$

$$\mathcal{L}^{-1}\left\{ \frac{-1}{s-2} + \frac{1}{s-3} \right\} = -e^{+2t} + e^{+3t} = e^{2t} * e^{3t}$$

(c) Find the function $f(t)$ which satisfies the equation

$$\int_0^t f(t-u)sin(u)du = t^2$$

$$\int_0^t f(t-u)\cdot\sin(u)\,du = f(t) * \sin t = t^2$$

Apply Laplace transform to both sides

$$\mathcal{L}\{f(t) * \sin t\} = \mathcal{L}\{t^2\}$$

$$F(s)\cdot\frac{1}{s^2+1} = \frac{2}{s^3} \Rightarrow F(s) = \frac{2s^2+2}{s^3} = 2\cdot\frac{1}{s} + \frac{2}{s^3}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = 2 + t^2$$

3.(8 + 8) Let $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ t-2 & 1 \le t < 2 \\ t^2 - 6t + 9 & 2 \le t \end{cases}$

(a) Write $f(t)$ as a combination of step functions $u_c(t)$.

$$f(t) = \underbrace{\underbrace{1 + u_1(t)(t-3)}_{t-2} + u_2(t)(t^2 - 7t + 11)}_{t^2-6t+9}$$

(b) Calculate the Laplace transform $\mathcal{L}\{f(t)\}$ by using your solution in part **(a)**.

$$f(t) = 1 + u_1(t)\left((t-1)-2\right) + u_2(t)\left((t-2)^2 - 3(t-2) + 1\right)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + e^{-s}\cdot\left(\frac{1}{s^2} - \frac{2}{s}\right) + e^{-2s}\left(\frac{2}{s^3} - 3\cdot\frac{1}{s^2} + \frac{1}{s}\right)$$

4.(20) Solve the initial value problem

$$y'' + 4y' + 5y = t + \delta(t-\pi) \quad y(0) = 1 \quad y'(0) = -1$$

Apply Laplace transform

$$s^2 Y(s) - s\underset{\substack{\parallel \\ 1}}{y(0)} - \underset{\substack{\parallel \\ -1}}{y'(0)} + 4sY(s) - 4\underset{\substack{\parallel \\ 1}}{y(0)} + 5Y(s) = \frac{1}{s^2} + e^{-\pi s}$$

$$Y(s)(s^2+4s+5) - s + 1 - 4 = \frac{1}{s^2} + e^{-\pi s}$$

$$Y(s)(s^2+4s+5) = \frac{1}{s^2} + e^{-\pi s} + s + 3$$

$$Y(s) = \frac{1}{s^2(s^2+4s+5)} + \frac{e^{-\pi s}\cdot 1}{s^2+4s+5} + \frac{s+3}{s^2+4s+5}$$

$$\frac{1}{s^2(s^2+4s+5)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+4s+5} = \frac{as^3+4as^2+5as+bs^2+4bs+5b+cs^3 \ldots}{s^2(s^2+4s+5)}$$

$$(s(s^2+4s+5)) \quad (s^2+4s+5) \quad (s^2)$$

$$\left.\begin{aligned} a+c &= 0 \\ 4a+b+d &= 0 \\ 5a+4b &= 0 \\ 5b &= 1 \end{aligned}\right\} \quad \begin{aligned} c &= \frac{4}{25} \\ d &= \frac{11}{25} \\ a &= -\frac{4}{25} \\ b &= \frac{1}{5} \end{aligned}$$

$$\frac{1}{s^2(s^2+4s+5)} = \frac{-\frac{4}{25}}{s} + \frac{\frac{1}{5}}{s^2} + \frac{\frac{1}{25}(4s+11)}{s^2+4s+5}$$

$$= \frac{-\frac{4}{25}}{s} + \frac{\frac{1}{5}}{s^2} + \frac{1}{25}\left(\frac{4(s+2)+3}{(s+2)^2+1^2}\right)$$

$$= \frac{-\frac{4}{25}}{s} + \frac{\frac{1}{5}}{s^2} + \frac{4}{25}\frac{(s+2)}{(s+2)^2+1^2} + \frac{3}{25}\frac{1}{(s+2)^2+ \ldots}$$

$$e^{-\pi s}\frac{1}{s^2+4s+5} = e^{-\pi s}\frac{1}{(s+2)^2+1^2}$$

$$\frac{s+3}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1^2} + \frac{1}{(s+2)^2+1^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-4}{25} + \frac{1}{5}\cdot t + \frac{4}{25}e^{-2t}\cdot\cos t + \frac{3}{25}e^{-2t}\sin t$$

$$+ U_\pi(t)\, e^{-2(t-\pi)}\cdot\sin(t-\pi) + e^{-2t}\cdot\cos t + e^{-2t}\cdot\sin t$$

5. (15) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\big((2-\lambda)(3-\lambda)\big) + 1\cdot(-(2$$

$$= (2-\lambda)\left((3-\lambda)^2 - 1\right) = (2-\lambda)(\lambda^2 - 6\lambda + 8) = (2-\lambda)^2\cdot(4-\lambda$$

$\underline{\underline{\lambda = 2}}$

$$(A - 2I)\cdot\vartheta = 0$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \vartheta_1 + \vartheta_3 &= 0 \\ \vartheta_1 &= -\vartheta_3 \\ \vartheta_2 &= \vartheta_2 \\ \vartheta_3 &= \vartheta_3 \end{aligned}$$

$$\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix} = \vartheta_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \vartheta_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = 4}}$

$$(A - 4I)\,\vartheta = 0$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{R_1+R_3} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left.\begin{aligned} -\vartheta_1 + \vartheta_3 &= 0 \\ -2\vartheta_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} \vartheta_1 &= \vartheta_3 \\ \vartheta_2 &= 0 \\ \vartheta_3 &= \vartheta_3 \end{aligned}$$

$$\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix} = \vartheta_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

6. (4+4+3+3) State whether the following statements are TRUE or FALSE, and justify your answer.

(a) If $f * g = g * f = f$ for any function f, then g is the Dirac delta function at zero, i.e. $\delta(t)$

<u>TRUE</u>:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \Rightarrow F(s) \cdot G(s) = F(s)$$

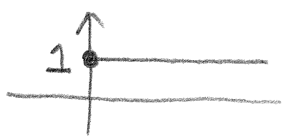
$$G(s) = 1$$

$$\mathcal{L}^{-1}\{G(s)\} = \delta(t) \checkmark$$

(b) If $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$, then $f(t) = g(t)$

<u>FALSE</u>:

$$f(t) = 1, \quad g(t) = \begin{cases} 1 & t \neq 1 \\ 2 & t = 1 \end{cases}$$



$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} g(t)\,d$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$$

but $f(t) \neq g(t)$

(c) If v_1 and v_2 are linearly independent vectors, then $A \cdot v_1$ and $A \cdot v_2$ will also be linearly independent for any matrix A.

<u>FALSE</u>: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A \cdot v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A \cdot v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are linearly dependent $\left(\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\right.$

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & -2 & -1 \\ -2 & 5 & -3 \end{bmatrix}$

<u>TRUE</u>

$$\begin{bmatrix} 1 & -4 & 3 \\ 3 & -2 & -1 \\ -2 & 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It's an eigenvector with eigenvalue 0.