M E T U
Northern Cyprus Campus

	Introduction to Differential Equations Midterm I	
Code : *Math 219* Acad.Year: *2010-2011* Semester : *Summer* Date : *19.7.2011* Time : *18:30* Duration : *120 minutes*	Last Name: Name : Department: Signature:	Student No: Section:
	7 QUESTIONS ON 6 PAGES TOTAL 105 POINTS	

1	2	3	4	5	6	7	

Show your work! No calculators! Please draw a box around your answers! Please do not write on your desk!

1. (10+5) Consider the initial value problem

$$\frac{dy}{dx} = x^2\sqrt[3]{y^2} \qquad y(1) = 1$$

(a) Find a solution for it.

Equation is seperable. $\displaystyle\int \frac{dy}{y^{2/3}} = \int x^2\,dx \qquad 3\cdot y^{1/3} = \frac{x^3}{3} + C$

$$y^{1/3} = \frac{x^3}{9} + \frac{C}{3} \Rightarrow y(x) = \left(\frac{x^3}{9} + C\right)^3$$

$$1 = y(1) = \left(\frac{1}{9} + C\right)^3 \xRightarrow{\text{taking cube root}} \frac{1}{9} + C = 1 \qquad C = \frac{8}{9}$$

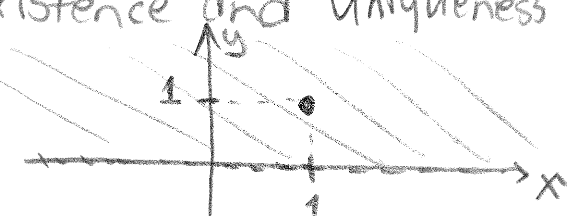
$$\therefore \boxed{y(x) = \left(\frac{x^3}{9} + \frac{8}{9}\right)^3}$$

(b) Is the solution unique ? Explain your answer.

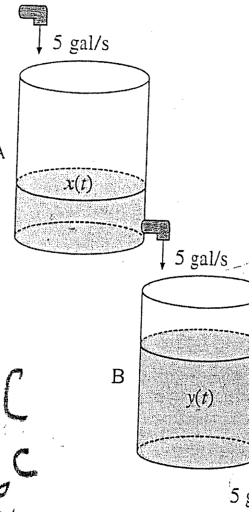
$f(x,y) = x^2\cdot\sqrt[3]{y^2}$ is continuous on $\mathbb{R}\times\mathbb{R}$

$\dfrac{\partial f}{\partial y} = \dfrac{2}{3}x^2\cdot\dfrac{1}{\sqrt[3]{y}}$ is continuous on $\mathbb{R}\times(\mathbb{R}\setminus\{0\})$

Hence $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ will be continuous on $\mathbb{R}\times(0,+\infty)$ containing $(1,1)$, then by Existence and Uniqueness Thm solution will be unique.



2. (4+6+4+6+[illegible]) Consider two tanks, A and B. Tank A contains 100 gal of solution in which is dissolved 20 kg of salt. Tank B contains 200 gal of solution in which is dissolved 40 kg of salt. Pure water enters in Tank A at a rate of 5 gal/s. Solution leaves Tank A with the same rate, 5 gal/s and enters immediately into Tank B. The drain at the bottom of Tank B let the solution to leave Tank B with the rate of 5 gal/s.



(a) Write a differential equation which models the amount of salt $x(t)$ in Tank A.

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out}) \Rightarrow \frac{dx}{dt} = 0 - \frac{x(t)}{100}\cdot 5$$

$$x(0) = 20 \qquad \frac{dx}{dt} = -\frac{1}{20}x$$

(b) Find $x(t)$ by solving the differential equation in part (a).

$$\frac{dx}{dt} = -\frac{1}{20}\cdot x \qquad \int \frac{dx}{x} = \int -\frac{1}{20}\,dt \qquad \ln|x(t)| = -\frac{1}{20}t + C$$

By taking the exponential, and knowing that $x(t) > 0$, $x(t) = e^{-\frac{1}{20}t}\cdot e^{C}$

$$20 = x(0) = e^{0}\cdot e^{C} \Rightarrow e^{C} = 20, \text{ so } x(t) = 20\cdot e^{-\frac{1}{20}t}$$

(c) Write a differential equation which models the amount of salt $y(t)$ in Tank B.

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} \Rightarrow \frac{dy}{dt} = \frac{x}{100}\cdot 5 - \frac{y}{200}\cdot 5 \qquad x(t) = 20\cdot e^{-\frac{1}{20}t}$$

$$\frac{dy}{dt} = e^{-t/20} - \frac{y}{40} \qquad y(0) = 40$$

(d) Find $y(t)$ by solving the differential equation in part (c).

linear: $\frac{dy}{dt} + \frac{y}{40} = e^{-t/20}$ $\qquad \mu(t) = e^{\int \frac{1}{40}dt} = e^{\frac{1}{40}t}$

$$\frac{d}{dt}\left(e^{\frac{1}{40}t}\cdot y\right) = e^{\frac{1}{40}t}\cdot e^{-\frac{1}{20}t} \Rightarrow e^{\frac{1}{40}t}\cdot y(t) = \int e^{-\frac{1}{40}t}\,dt = -40\,e^{-\frac{1}{40}t} + C$$

$$y(t) = -40\cdot e^{-\frac{1}{20}t} + C\cdot e^{-\frac{1}{40}t}$$

$$40 = y(0) = -40 + C \Rightarrow C = 80 \qquad \boxed{y(t) = -40\cdot e^{-\frac{1}{20}t} + 80\cdot e^{-\frac{1}{40}t}}$$

3)(5+5+5) Consider the differential equation

$$(x^4 - y)dx + x(xy^2 + 1)dy = 0$$

(a) Show that this differential equation is not exact.

$M(x,y) = x^4 - y \quad N(x,y) = x(xy^2+1)$

$M_y = -1 \quad N_x = 2xy^2 + 1, \quad M_y \neq N_x$, so "not Exact"

(b) Find an integrating factor $\mu(x)$ which makes the differential equation exact.

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \cdot \mu \Rightarrow \frac{d\mu}{dx} = \frac{-2 - 2xy^2}{x \cdot (xy^2+1)} \cdot \mu = \frac{-2\cancel{(1+x \cdot y^2)}}{x \cdot \cancel{(xy^2+1)}} \cdot$$

$$\frac{d\mu}{dx} = -\frac{2}{x} \cdot \mu \Rightarrow \int \frac{d\mu}{\mu} = \int -\frac{2}{x}\, dx \quad \ln|\mu| = -2\ln|x|$$

$$\ln|\mu| = \ln|x^{-2}| \Rightarrow \mu(x) = x^{-2}$$

(c) By using the integrating factor found in **(b)**, solve the differential equation.

Multiply the equation by $\mu(x) = x^{-2} = \frac{1}{x^2}$

$$\underbrace{\left(x^2 - \frac{y}{x^2}\right)}_{M(x,y)} dx + \underbrace{\left(y^2 + \frac{1}{x}\right)}_{N(x,y)} dy = 0$$

$M_y = -\frac{1}{x^2} \quad N_x = -\frac{1}{x^2}$, so $M_y = N_x$, it's exact now

Then, there exists $F(x,y)$ such that $\frac{\partial F}{\partial x} = x^2 - \frac{y}{x^2}$

$$\frac{\partial F}{\partial y} = y^2 + \frac{1}{x}$$

$$F(x,y) = \int x^2 - \frac{y}{x^2}\, dx = \frac{x^3}{3} + \frac{y}{x} + h(y)$$

$$\frac{\partial F}{\partial y} = \frac{1}{x} + h'(y) = \frac{1}{x} + y^2 \Rightarrow h'(y) = y^2 \quad h(y) = \frac{y^3}{3} + C$$

$\therefore \frac{x^3}{3} + \frac{y}{x} + \frac{y^3}{3} = C$ is the solution.

4. (10+10) Find the general solution to the following non-homogeneous differential equations by using the method of undetermined coefficients.

(a) $y'' + 2y' - 3y = 3t$

$$r^2 + 2r - 3 = (r+3)(r-1) \Rightarrow r_1 = -3,\ r_2 = 1$$

$$y_h(t) = c_1 \cdot e^{-3t} + c_2 \cdot e^{t}$$

Trial: $y_p(t) = At + B \qquad y_p'(t) = A \qquad y_p''(t) = 0$

$$0 + 2 \cdot A - 3At - 3B = 3t$$

$$-3A = 3 \qquad 2A - 3B = 0$$

$$A = -1 \qquad B = -\frac{2}{3}$$

$$y_g(t) = c_1 \cdot e^{-3t} + c_2 \cdot e^{t} - t - \frac{2}{3}$$

(b) $y'' + 2y' + y = -2e^{-t}$

$$r^2 + 2r + 1 = (r+1)^2 \Rightarrow r_1 = r_2 = -1$$

$$y_h(t) = c_1 \cdot e^{-t} + c_2 \cdot t \cdot e^{-t}$$

Trial: $y_p(t) = At^2 \cdot e^{-t} \qquad y_p'(t) = 2At \cdot e^{-t} - At^2 \cdot e^{-t}$

$$y_p''(t) = 2A \cdot e^{-t} - 2At \cdot e^{-t} - 2At \cdot e^{-t} + A \cdot t^2 \cdot e^{-t}$$

$$2A \cdot e^{-t} \cancel{- 4At \cdot e^{-t}} \cancel{+ A \cdot t^2 \cdot e^{-t}} \cancel{+ 4At \cdot e^{-t}} \cancel{- 2At^2 \cdot e^{-t}} \cancel{+ At^2 \cdot e^{-t}} = -2$$

$$2A = -2 \Rightarrow A = -1$$

$$\boxed{y_g(t) = c_1 \cdot e^{-t} + c_2 \cdot t \cdot e^{-t} - t^2 \cdot e^{-t}}$$

5. (4+8+8) The parts below are about the non-homogeneous differential equation

$$t^2y'' - ty' + y = t^4$$

(a) Without finding any solution, find the largest interval where the solution with initial value $y(1) = a,\ y'(1) = b$ exists and is unique.

$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = t^2 \qquad p(t) = -\frac{1}{t},\ q(t) = \frac{1}{t^2},\ g(t) = t^2$

are all continuous on $(0, +\infty)$ containing $t_0 = 1$

$\therefore$ $(0, +\infty)$ is biggest interval where solution exists and is unique b

Existence and Uniqueness Thm.

(b) If $y_1(t) = t$ is a solution to the associated homogeneous equation, find the second fundamental solution $y_2(t)$ by using **reduction of order method**.

$y_2(t) = \vartheta \cdot t \quad y_2'(t) = \vartheta' \cdot t + \vartheta \quad y_2''(t) = \vartheta'' \cdot t + 2\vartheta'$

$t^2(\vartheta'' \cdot t + 2\vartheta') - t \cdot (\vartheta' \cdot t + \vartheta) + \vartheta \cdot t = 0$

$t^3 \cdot \vartheta'' + (2t^2 - t^2) \cdot \vartheta' - \cancel{\vartheta \cdot t} + \cancel{\vartheta \cdot t} = 0$

$t^3 \cdot \vartheta'' + t^2 \vartheta' = 0 \qquad u = \vartheta' \quad u' = \vartheta'' \qquad t > 0$

$u' + \frac{1}{t}u = 0 \qquad \frac{du}{dt} = -\frac{1}{t}u$

$\int \frac{du}{u} = \int -\frac{1}{t}\,dt \qquad \ln|u| = \ln|t^{-1}| \qquad u = \frac{1}{t} \qquad \vartheta = \ln t + C$

$y_2(t) = t \cdot \ln t + \cancel{C}t$

(c) Use variation of parameters to find the general solution to the non-homogenous equation.

$W(y_1, y_2) = \begin{vmatrix} t & t \cdot \ln t \\ 1 & \ln t + 1 \end{vmatrix} = t \cdot \ln t + t - t \cdot \ln t = t \neq 0 \quad \underline{\underline{t>}}$

$u_1 = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)}\,dt = -\int \frac{\cancel{t} \cdot \ln t \cdot t^2}{\cancel{t}}\,dt = -\int t^2 \cdot \ln t\,dt$

$u = \ln t \quad du = \frac{1}{t}dt$

$dv = t^2\,dt \quad v = \frac{t^3}{3}$

$= -\frac{t^3}{3} \cdot \ln t + \int \frac{t^2}{3}\,dt = -\frac{t^3}{3} \cdot \ln t + \frac{t^3}{9} + C$

$u_2 = \int \frac{y_1(t) \cdot g(t)}{W(y_1,y_2)}\,dt = \int \frac{\cancel{t} \cdot t^2}{\cancel{t}}\,dt = \frac{t^3}{3} + C$

$y_g(t) = C_1 \cdot t + C_2 \cdot t \cdot \ln t + \left(-\frac{t^3}{3} \cdot \ln t + \frac{t^3}{9}\right) \cdot t + \left(\frac{t^3}{3}\right) \cdot t \cdot \ln t$

Choose one of the following

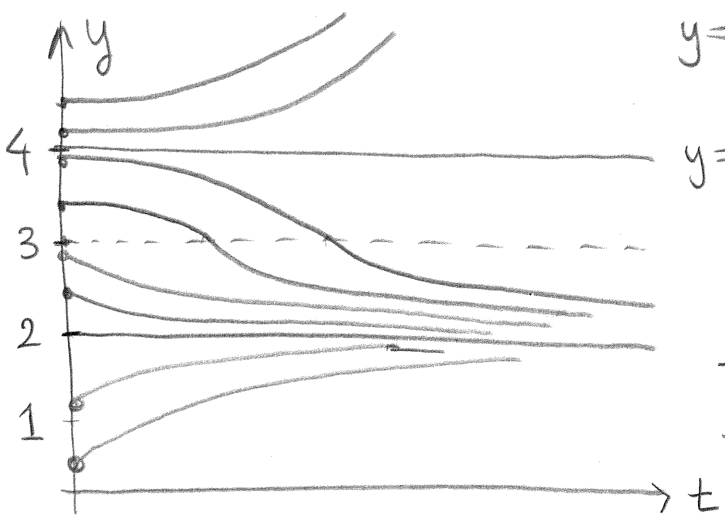
6. (15) For the autonomous differential equation,

$$\frac{dy}{dt} = y^2 - 6y + 8 \qquad y > 0$$

draw the integral curves, determine the equilibrium solutions, and describe their stabilities.

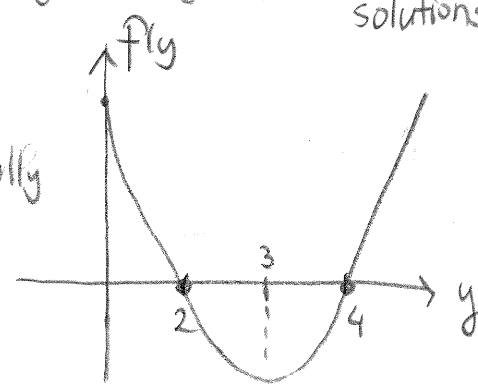
$\frac{dy}{dt} = f(y) = y^2 - 6y + 8 = (y-2)\cdot(y-4) = 0 \Rightarrow y = 2,\ y = 4$ are equilibri[um] solutions

$\frac{d^2y}{dt^2} = f'(y)\cdot f(y) = (2y-6)\cdot(y-2)\cdot(y-4)$



$y=2$ Asymptotically Stable

$y=4$ Unstable



	$0<y<2$	$2<y<3$	$3<y<4$	$4<y$
$f(y)$	+	−	−	+
$f'(y)$	−	−	+	+
	increasing concave down	decreasing concave up	decreasing concave down	increasing concave up

7. (15) Consider the damped free spring mass system which is described by the differential equation

$$mu'' + \gamma u' + ku = 0$$

where $u(t)$ is the displacement of the mass from the equilibrium position.
Show that if the system is critically damped, then the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
(*Hint:* At the equilibrium position, u(t)=0)

If the system is critically damped, then $\gamma = 2\sqrt{km}$

i.e. characteristic polynomial $m\cdot r^2 + \gamma\cdot r + k$ will have repeated roots $r_{1,2} = \frac{-\gamma}{2m}$ since $\Delta = \gamma^2 - 4km = 0$

Hence, the solution to $u(t) = (A+Bt)\cdot e^{\frac{-\gamma}{2m}t}$

Since $e^{\frac{-\gamma}{2m}t} \neq 0$, $u(t) = 0$ implies $A + Bt = 0$

which can happen at most once when $t = -\frac{A}{B}$ $B \neq 0$

If $B = 0$ $u(t) \neq 0$