METU - NCC

	Differential Equations Midterm II	
Code : *Math 219* Acad.Year: *2011-2012* Semester : *Spring* Date : *29.3.2011* Time : *17:40* Duration : *120 minutes*	Last Name: Name : KEY Department: Signature:	Student No.: Section:
	6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	

1 (15)	2 (20)	3 (10)	4 (15)	5 (20)	6 (20)		

1. (5+10pts) Given the differential equation

$$(2yxe^x + 2x^2ye^x)dx + (6y^3x - 2x^2e^x)dy = 0$$

a. Show that $\mu = \dfrac{1}{xy^2}$ is an integrating factor.

$$\underbrace{\left(\frac{e^x}{y} + \frac{xe^x}{y}\right)}_{M} dx + \underbrace{\left(3y - \frac{xe^x}{y^2}\right)}_{N} dy = 0$$

$$M_y = -\frac{e^x}{y^2} - \frac{xe^x}{y^2} \overset{!}{=} N_x = -\frac{e^x}{y^2} - \frac{xe^x}{y^2}$$

b. Find the general solution to the differential equation.

There is a potential function $\Psi(x,y)$ s.t.

$$\left.\begin{array}{l} \Psi_x = \dfrac{e^x}{y} + \dfrac{xe^x}{y} \\ \Psi_y = 3y - \dfrac{xe^x}{y^2} \end{array}\right\} \Rightarrow \Psi = \frac{e^x}{y} + \frac{(x-1)e^x}{y} + C(y) = \frac{xe^x}{y} + C(y)$$

$$\Downarrow$$

$$= -\frac{xe^x}{y^2} + C'(y)$$

Therefore $C'(y) = 3y \Rightarrow C(y) = \frac{3}{2}y^2 + C$

Whence

$$\Psi(x,y) = \frac{xe^x}{y} + \frac{3}{2}y^2 = \text{const}$$

is an implicitly defined family of solutions.

2. (5+15 pts) Consider the differential equation

$$ty'' - (2t+1)y' + 2y = 0\,, \quad t>0$$

a. Show that the function $y_1 = e^{2t}$ is a solution.

$y_1' = 2e^{2t}$, $y_1'' = 4e^{2t}$

$4te^{2t} - (2t+1)2e^{2t} + 2e^{2t} = 0$, that is,

$y_1 = e^{2t}$ is a solution to.

b. Using the **reduction of order method**, find the general solution.

Put $y = e^{2t}\cdot v$. Then $y' = 2e^{2t}\cdot v + e^{2t}\cdot v'$ and

$y'' = 4e^{2t}v + 4e^{2t}v' + e^{2t}v''$. It follows that

$te^{2t}v'' + (4te^{2t} - (2t+1)e^{2t})v' + (4t - 2(2t+1) + 2)e^{2t}v = 0$,

$te^{2t}v'' + (2t-1)e^{2t}v' = 0$,

$tv'' + (2t-1)v' = 0$ (a separable dif. eq. w.r.t. v')

$\frac{dv'}{v'} = \frac{1-2t}{t}dt \Rightarrow \ln|v'| = \ln(t) - 2t + C \Rightarrow$

$v' = cte^{-2t} \Rightarrow v = c\int te^{-2t}dt + K =$

$= c\left(-\frac{1}{2}\left(te^{-2t} - \int e^{-2t}dt\right)\right) + K =$

$= c\left(-\frac{t}{2}e^{-2t} - \frac{e^{-2t}}{4}\right) + K = -\frac{c}{4}(2t+1)e^{-2t} + K$

We can assume that $K=0$ and $c=-4$. Thus

$y_2 = (2t+1)e^{-2t}e^{2t} = 2t+1$ and

$y = C_1e^{2t} + C_2(2t+1)$ is the general solution to.

$W(t) = \begin{vmatrix} e^{2t} & 2t+1 \\ 2e^{2t} & 2 \end{vmatrix} = -2te^{2t} \neq 0$, for $t>0$.

3. (10pts) Solve the following initial value problem:

$$y'' + 2y' + 5y = 0, \qquad \text{with } y(0) = 0, \quad y'(0) = 2.$$

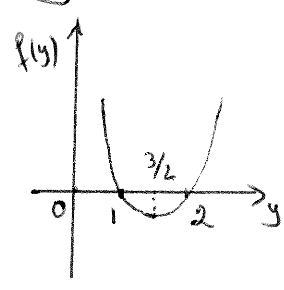
The characteristic equation $r^2 + 2r + 5 = 0$ has the negative discriminant $\Delta = -16$. It has complex conjugate tools $r = -1 \pm i2$. Hence $y = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$ is the general solution.

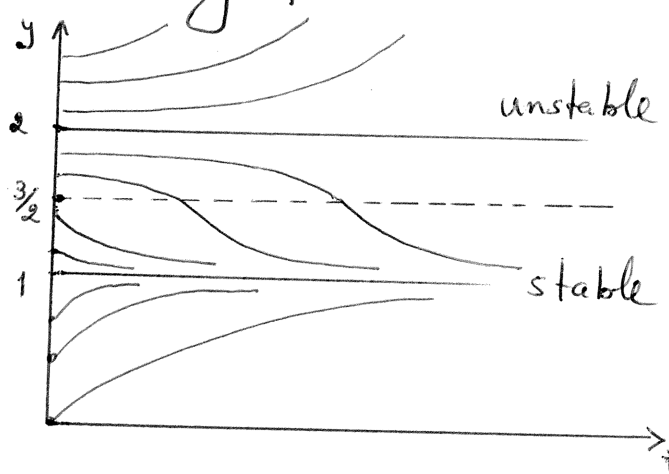
But $0 = y(0) = C_1 \Rightarrow y = C_2 e^{-t}\sin(2t)$

$2 = y'(0) = -C_2 e^{-t}\sin(2t) + 2C_2 e^{-t}\cos(2t)\big|_{t=0} = 2C_2$, that is, $C_2 = 1$. Whence $y = e^{-t}\sin(2t)$ is the solution to IVP.

4. (15pts) For the autonomous differential eqution $y' = y^2 - 3y + 2$, $y \geq 0$, draw the integral curves. Mark the equilibrium solutions, and label them as asymptotically stable, unstable, or semi-stable.

$y' = (y-1)(y-2) \Rightarrow y = 1$ and $y = 2$ are equilibrium solutions. Note that $f(y) = (y-1)(y-2)$ is a parabola looked up with its minimal value at $y = \frac{3}{2}$; $y'' = f'(y)f(y)$ controls the concavity of solutions





	$0 \le y \le 1$	$1 \le y \le 3/2$	$3/2 \le y \le 2$	$y \geq 2$
$y' = f(y)$	+	−	−	+
$y'' = f'(y)f(y)$	−	+	−	+
y	incr. c. down	decr. c. up	decr. c. down	incr. c. up

5. (16+4pts) A tank has 10 gallons of water which contains 2 kg of dissolved salt. Water with a varying concentration of $\frac{t}{4}$ kg of salt per gal is pumped into the tank at the rate of 2 gal/min, and the mixture - kept uniform by stirring - is pumped out at the same rate. Let $Q(t)$ be the amount of salt in kg in the tank at time t (t in terms of min).

a. Find $Q(t)$. We have $Q'(t) = $ rate in $-$ rate out $=$

$$= \frac{t}{4} \cdot 2 - \frac{Q(t)}{10} \cdot 2 \Rightarrow Q'(t) + \frac{1}{5} Q(t) = \frac{t}{2}$$

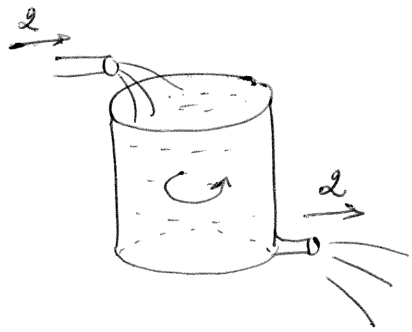
$\mu(t) = e^{t/5}$ is an integrating factor. Then

$$(\mu \cdot Q)' = \frac{t}{2} e^{t/5} \Rightarrow \mu Q = \frac{1}{2} \int t e^{t/5} dt + C =$$

$$= \left(\frac{5}{2} t - \frac{25}{2}\right) e^{t/5} + C \Rightarrow Q(t) = \frac{5}{2} t - \frac{25}{2} + C e^{-t/5}.$$

But $Q(0) = 2$ kg. Therefore $2 = C - \frac{25}{2} \Rightarrow C = \frac{29}{2}$.

Hence $Q(t) = \frac{5}{2} t - \frac{25}{2} + \frac{29}{2} e^{-t/25}$ is the amount of salt in the tank at time t.



b. Does $Q(t)$ exceed 8 kg when $t \leqslant 5$? Explain.

Since $Q(t)$ is an increasing function, we have to look at the amount at $t = 5$. But

$$Q(5) = \frac{25}{2} - \frac{25}{2} + \frac{29}{2} \frac{1}{e} = \frac{29}{2 \cdot e} < 8 \text{ kg}.$$

Thus $Q(t)$ doesn't exceed 8 kg when $t \leq 5$.

6. (5+15pts) Consider the non-homogeneous differential equation

$$y'' + \overset{4}{\cancel{2}}y = 2\sin(2t) + te^{-t}$$

a. Give the solution to the corresponding homogenous differential equation.

$r^2 + 4 = 0 \Rightarrow r = \pm i2$ roots of the char. eq.

$y = C_1\cos(2t) + C_2\sin(2t)$ is the general solution to homg. dif. eq.

b. Use the **method of undetermined coefficients** to find the general solulion.

We have two subproblems $y'' + 4y = 2\sin(2t)$ and $y'' + 4y = te^{-t}$.

Put $Y_1(t) = t(A\cos(2t) + B\sin(2t))$ (duplication in)

$4B\cos(2t) - 4A\sin(2t) - 4At\cos(2t) - 4Bt\sin(2t) +$

$+ 4tA\cos(2t) + 4Bt\sin(2t) = 2\sin(2t)$ or

$4B\cos(2t) - 4A\sin(2t) = 2\sin(2t)$

$B = 0,\ A = -\frac{1}{2} \Rightarrow Y_1(t) = -\frac{1}{2}t\cos(2t)$

Put $Y_2(t) = (At + B)e^{-t}$ (no duplication in)

$-2Ae^{-t} + Be^{-t} + Ate^{-t} + 4(At + B)e^{-t} = te^{-t}$ or

$(-2A + 5B) + 5At = t$

$A = \frac{1}{5},\ B = \frac{2}{25} \Rightarrow Y_2(t) = \left(\frac{t}{5} + \frac{2}{25}\right)e^{-t}$

Thereby

$y = C_1\cos(2t) + C_2\sin(2t) - \frac{1}{2}t\cos(2t) + \left(\frac{t}{5} + \frac{2}{25}\right)e^{-t}$

is the general solution to nonhomg. dif. eq.